University of Delaware
Department of Mathematical Sciences
Math 810 Asymptotic and Perturbation Methods 09S
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Homework 6: Due 4/29/09

1. Fill in the details of the linear stability analysis of the periodic solutions of the damped and driven Duffing equation discussed in class (also in §3.4 of Holmes and §4.3.1 in Kevorkian and Cole). Consider $\beta = 0$ and $\beta \neq 0$ for the nonlinear case (see p. 314-5 for more detail).

2. Consider the integral
$$I(\sigma) = \int_0^1 e^{\sigma t^3}, \quad |\sigma| \to \infty > 0,$$
in the complex t-plane.
(a) Compute an asymptotic approximation with all terms with Re $\sigma > 0$. Do this by converting the integral into two integrals with the limits of integration from $-\infty$ to 1 and from $-\infty$ to 0. Use Watson's lemma for the first integral and the definition of the Γ function for the second to obtain
$$I(i\nu) \sim -\frac{1}{\sigma^{1/3}}\Gamma\left(\frac{4}{3}\right) + \frac{e^\sigma}{3}\sum_{n=0}^{\infty}\frac{\Gamma(n+2/3)}{\Gamma(2/3)}\,\frac{1}{\sigma^{n+1}}$$
(b) Why is this result restricted to Re $\sigma > 0$?.
(c) Reexamine I for $\sigma = -\nu$ and Re $\nu > 0$. Now use two integrals from 0 to ∞ and from 1 to ∞. Use the Γ function definition for the first and Watson's lemma for the second to get a new representation for $|\arg \nu| < \pi/2$.
(d) We still don't have an asymptotic approximation for the integral on the imaginary axis. Use the method of steepest descent to get an approximation for this case. Let $\sigma = i\nu$ with Re $\nu > 0$; then
$$I(i\nu) = \int_0^1 e^{i\nu z^3} dz$$
where z is complex. Find the steepest curves through $z = 0$ and $z = 1$ and choose appropriate inegration paths to find
$$I(i\nu) \sim \frac{e^{i\pi/6}}{\nu^{1/3}}\Gamma\left(\frac{4}{3}\right) + \frac{1}{3}e^{i\nu}\sum_{n=0}^{\infty}\frac{\Gamma(n+2/3)}{\Gamma(2/3)}\,\frac{1}{(i\nu)^{n+1}}$$
for $|\arg \nu| < \pi/2$.
(e) Do similar things for $\sigma = -i\nu$, Re $\nu > 0$. In this way we can find approximations for all arg σ; the changes of the formulas across the imaginary axis are an example of Stokes phenomenon.

3. Using the WKB method, recover the leading order uniformly valid asymptotic approximation of the solution that was obtained from boundary layer theory for the following problem.
$$\epsilon y'' + a(x)y' + b(x)y = 0, \;\; y(0) = A, \;\; y(1) = B,$$
with $a(x) > 0$ on $0 \le x \le 1$ for $\epsilon \ll 1$. Note that you will have to neglect the TST $\exp[-(1/\epsilon)\int_0^1 a(t)dt]$ in order to recover the boundary layer approximation, and that you will not need matching to get this solution.