Math 428 Numerical and Algorithmic Solution of Differential Equations 08S
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Your name: ____

Project 4. Due 5/21/08, 4 pm.
In this project you will examine the numerical solution of the initial boundary value problem

$$u_t = u_{xx} + f_\ell(x),\ u(0,t) = u(1,t) = 0,\ u(x,0) = -\sin(\pi x).$$

Here $\ell = 1, 2$; when $\ell = 1$, $f_1(x) = 0$ and when $\ell = 2$, $f_2(x) = 3\pi^2 \sin(3\pi x)$. Let $u_\ell(x,t)$ be the solution for that value of ℓ. You will find the solution by applying the Crank-Nicolson method. The point of this project is to see the different error behaviors for an unconditionally stable method.

1. Show by substitution that the exact solution to the IBVP is

   $$u_1(x,t) = -e^{-\pi^2 t} \sin(\pi x)$$

   for $\ell = 1$, and when $\ell = 2$, the exact solution is given by

   $$u_2(x,t) = u_1(x,t) + \frac{1}{3}\left(1 - e^{-9\pi^2 t}\right) \sin(3\pi x).$$

   Note that $u_1(x,t)$ is a transient solution that comes from the initial conditions, while the second part of u_2 will, in the limit of large t, give the steady state solution from $f_2(x)$.

2. Develop a code for solving the IBVP using the Crank-Nicolson method. The method for this problem is

   $$A\mathbf{w}_{j+1} = B\mathbf{w}_j = \mathbf{b}$$

   where the matrix A has elements

   $$a_{i,i} = 1 + \lambda$$

   on the diagonal and

   $$a_{i-1,i} = a_{i,i+1} = -\lambda/2$$

   on the sub- and super-diagonals and the "right-hand-side" vector has elements

   $$b_i = (1-\lambda)w_{i,j} + (\lambda/2)(w_{i-1,j} + w_{i+1,j}) + kf_\ell(x_i)$$

   for the grid points $x_i = ih$, $i = 1, 2, \ldots, M-1$ and $w_{i,j} \approx u(x_i, t_j)$. You have to solve this tridiagonal system at each time step, and repeat the process until the desired time is reached.

3. Define $h = 1/N_x$ with $N_x = 350$ for the spatial grid step and run your code with $k = 1/N_t$ with $N_t = 4, 8, 16, 32, 64, 128$ for the Crank-Nicolson method.

4. Plot the solution as a function of x for $t = 0, 1.0$, and some representative time levels in between, with $N = 64$ on a single plot.

5. Compute the infinity norm of the relative error for each k at $t = 0.5$ and plot them on a log-log plot *vs.* k. What is the order of convergence with k for the method?

6. Compute the infinity norm of the relative error for $N_t = 512$ and for $N_x = 8, 16, 32, 64, 128, 256, 512$ and plot the resulting values on a log-log plot again. What is the order of convergence in h?

Hand in your plots, demonstration of the exact solution (handwritten is fine for this part), two pages or less of discussion that adequately discusses your results (so that you can pick it up in a year and know what problem you did and what the results were), and this sheet as a cover sheet.

The project is individual (outside of discussing the general approach you're using or want to use); sharing of code is not permitted.