

# A Overset Grid Method for Fourth Order Evolution Equations of Human Tear Film

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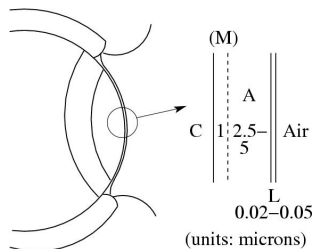
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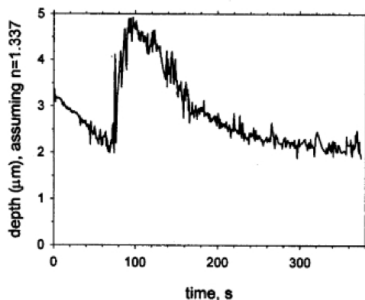
# Human Tear Film

**Tear Film:** A multilayer structure playing a vital role in health and function of the eye.



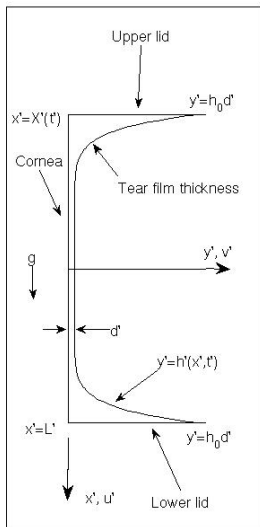
Typical thickness of each layer in microns.

Taken from King-Smith *et al.* 2000.



**Goal:** Simulate the effects of reflex tearing.

# Idealized domain



Characteristic scales:

$L' = 5\text{mm}$ , half-length of cornea;

$d' = 5\mu\text{m}$ , thickness of film;

$U_m = 10 - 30\text{ cm/s}$ , velocity;

$L'/U_m = .05\text{s}$ , time scale;

$\mu U_m / (d' \epsilon)$ , pressure scale.

The ratio of length scales

$\epsilon = d'/L' \approx 10^{-3} \Rightarrow$  lubrication.

# Formulation

At leading order, on  $0 \leq y \leq h(x, t)$  and  $X(t) \leq x \leq 1$  we have

$$u_x + v_y = 0, \quad u_{yy} - p_x + G = 0 \quad \text{and} \quad p_y = 0.$$

**Boundary conditions:**

**The wall:** Navier slip condition and impermeability

$$u = \beta u_y, \quad v = 0.$$

**Free surface:** Kinematic condition:  $h_t + uh_x = v - E$ .

Normal stress condition:  $p = -Sh_{xx}$ .

Strong insoluble surfactant: uniform stretching

$$\text{limit, } u(x, h(x, t), t) = X_t \frac{1-x}{1-X}.$$

The **evolution of the free surface** is

$$h_t + \left[ \frac{h^3}{12} \left( 1 + \frac{3\beta}{h+\beta} \right) (Sh_{xxx} + G) + X_t \frac{1-x}{1-X} \frac{h}{2} \left( 1 + \frac{\beta}{h+\beta} \right) \right]_x + E = 0$$

where  $\beta = \frac{L_s}{d^2}$ ,  $G = \frac{\rho g d^2}{\mu U_m}$ ,  $E = \frac{J'}{U_m \epsilon \rho}$  and  $S = \frac{\epsilon^3 \sigma}{\mu U_m}$ .

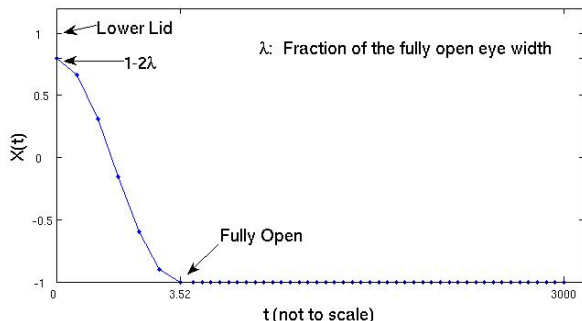
# Blink Cycle and Lid Motion

A **blink cycle** in reflex tear study includes:

**Upstroke:** Opening of the lids, 0.1758s.

**Interblink:** Lids remain open, 25 – 150s.

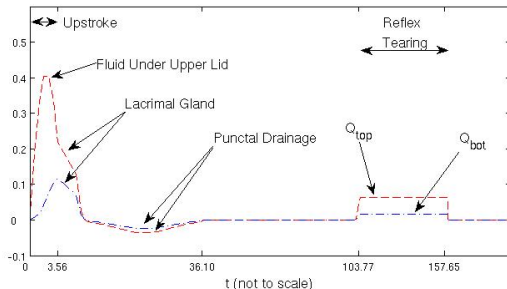
**Lid motion:** Many blinks are incomplete (Doane): lids don't completely close. Model for upper lid motion,  $X(t)$ :



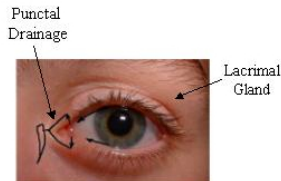
(Berke & Mueller, Jones *et al.*, Heryudono *et al.*)

# Flux Boundary Conditions

One-dimensional approximations:



**Reflex Tearing:** Excess tear fluid from lacrimal gland resulting from external or internal stimuli.

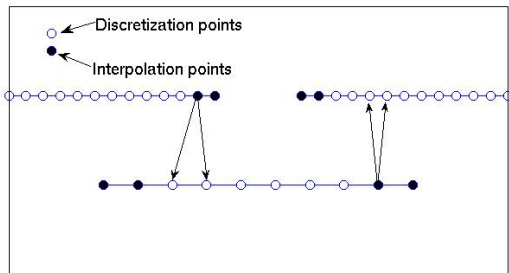


# Overset Grid Method

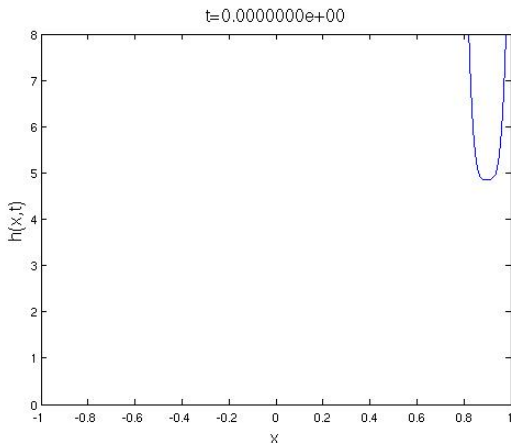
Map moving domain  $X(t) \leq x \leq 1$  into the **fixed domain**  $-1 \leq \xi \leq 1$

$$\xi = 1 - 2(1 - x)/(1 - X(t)).$$

Implemented a finite difference based method of lines on a overset grid. An example of a **overset grid**:



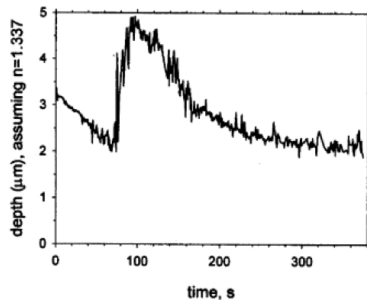
# Typical Reflex Tearing Calculation



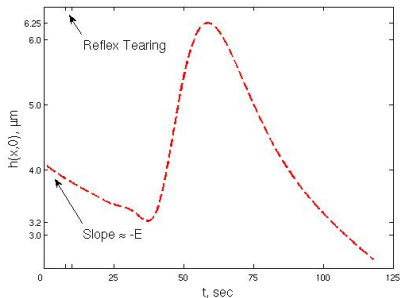
Parameters:  $h_0 = 13$ ,  $\beta = 10^{-3}$ ,  $S = 9e^{-7}$ ,  $G = 2.5e^{-3}$ ,  $E = 3e^{-4}$ ,  
 $h_e = 0.6$ ,  $\lambda = 0.1$ .

# Comparison with *In Vivo* Measurements

Measured data taken from King-Smith *et al.* 2000.



Computation from model.



Single location measurement from the center of the cornea.

# Conclusions

## Summary:

Qualitative agreement with King-Smith *et al.* measured data.

Mechanism for initial decrease is evaporation.

Mechanism for jump is drainage from upper meniscus by gravity.

Reflex tearing can break through the black lines.

## Future work:

Capture leveling off in King-Smith *et al.*'s measurement.

The inclusion of van der Waals forces.

Two-dimensional simulations with Overture framework developed at Lawrence Livermore National Laboratory.