A Overset Grid Method for Fourth Order Evolution Equations of Human Tear Film

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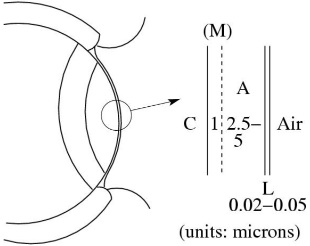
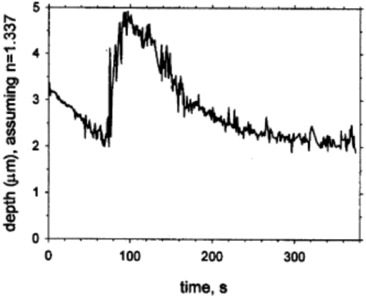
Human Tear Film

Tear Film: A multilayer structure playing a vital role in health and function of the eye.



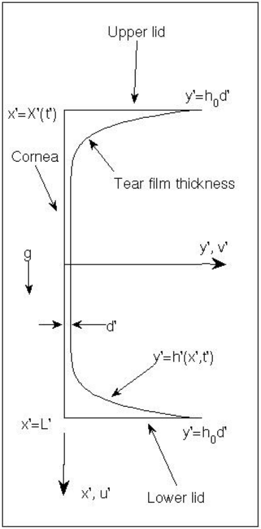
Typical thickness of each layer in microns.

Taken from King-Smith *et al.* 2000.



Goal: Simulate the effects of reflex tearing.

Idealized domain



Characteristic scales:

$L' = 5mm$, half-length of cornea;
$d' = 5\mu$m, thickness of film;
$U_m = 10 - 30$ cm/s, velocity;
$L'/U_m = .05$s, time scale;
$\mu U_m/(d'\epsilon)$, pressure scale.

The ratio of length scales
$\epsilon = d'/L' \approx 10^{-3} \Rightarrow$ lubrication.

Formulation

At leading order, on $0 \leq y \leq h(x,t)$ and $X(t) \leq x \leq 1$ we have

$$u_x + v_y = 0, \quad u_{yy} - p_x + G = 0 \;\text{ and }\; p_y = 0.$$

Boundary conditions:

The wall: Navier slip condition and impermeability
$u = \beta u_y$, $v = 0$.

Free surface: Kinematic condition: $h_t + uh_x = v - E$.
Normal stress condition: $p = -Sh_{xx}$.
Strong insoluble surfactant: uniform stretching limit, $u(x, h(x,t), t) = X_t \frac{1-x}{1-X}$.

The evolution of the free surface is

$$h_t + \left[\frac{h^3}{12}\left(1 + \frac{3\beta}{h+\beta}\right)(Sh_{xxx} + G) + X_t \frac{1-x}{1-X}\frac{h}{2}\left(1 + \frac{\beta}{h+\beta}\right)\right]_x + E = 0$$

where $\beta = \frac{L_s}{d'}$, $G = \frac{\rho g d'^2}{\mu U_m}$, $E = \frac{J'}{U_m \epsilon \rho}$ and $S = \frac{\epsilon^3 \sigma}{\mu U_m}$.

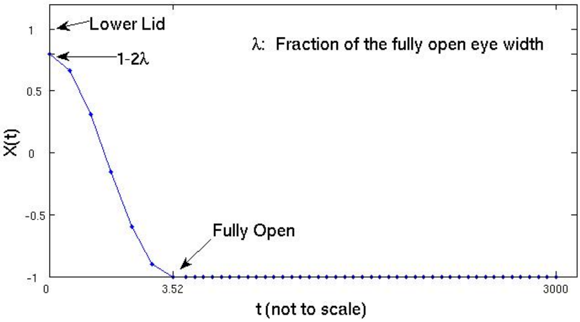
Blink Cycle and Lid Motion

A blink cycle in reflex tear study includes:

Upstroke: Opening of the lids, 0.1758s.

Interblink: Lids remain open, 25 – 150s.

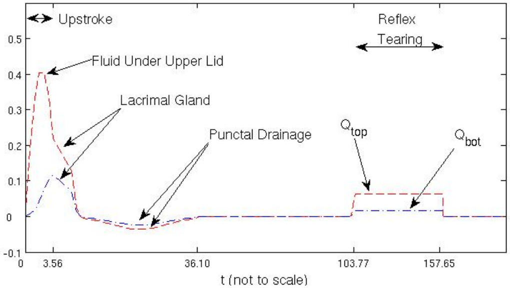
Lid motion: Many blinks are incomplete (Doane): lids don't completely close. Model for upper lid motion, $X(t)$:

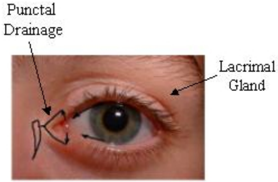


(Berke & Mueller, Jones *et al.*, Heryudono *et al.*)

Flux Boundary Conditions

One-dimensional approximations:





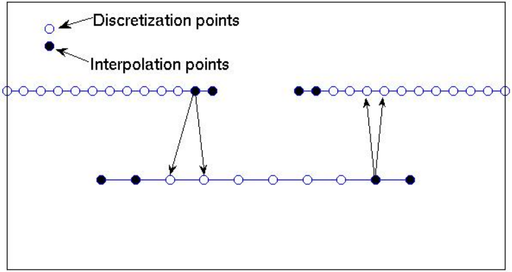
Reflex Tearing: Excess tear fluid from lacrimal gland resulting from external or internal stimuli.

Overset Grid Method

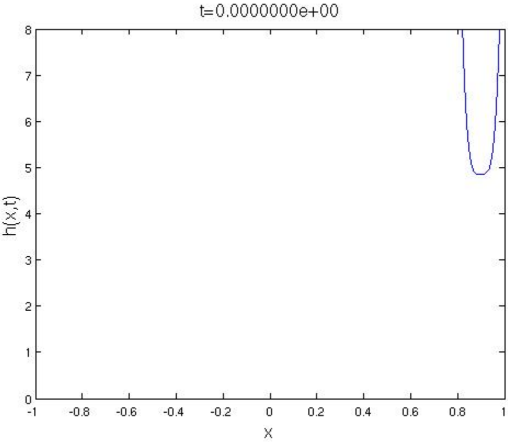
Map moving domain $X(t) \leq x \leq 1$ into the fixed domain $-1 \leq \xi \leq 1$

$$\xi = 1 - 2(1 - x)/(1 - X(t)).$$

Implemented a finite difference based method of lines on a overset grid. An example of a overset grid:



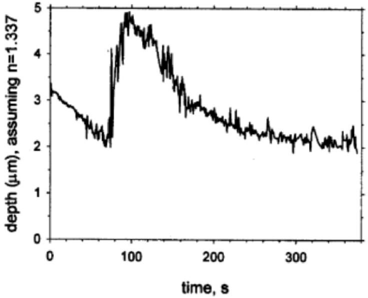
Typical Reflex Tearing Calculation



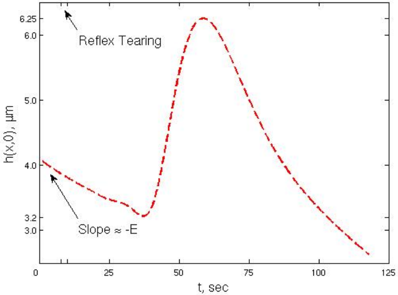
Parameters: $h_0 = 13$, $\beta = 10^{-3}$, $S = 9e^{-7}$, $G = 2.5e^{-3}$, $E = 3e^{-4}$, $h_e = 0.6$, $\lambda = 0.1$.

Comparison with *In Vivo* Measurements

Measured data taken from King-Smith *et al.* 2000.



Computation from model.



Single location measurement from the center of the cornea.

Conclusions

Summary:

- Qualitative agreement with King-Smith *et al.* measured data.
- Mechanism for initial decrease is evaporation.
- Mechanism for jump is drainage from upper meniscus by gravity.
- Reflex tearing can break through the black lines.

Future work:

- Capture leveling off in King-Smith *et al.*'s measurement.
- The inclusion of van der Waals forces.
- Two-dimensional simulations with Overture framework developed at Lawrence Livermore National Laboratory.