Human Tear Film Dynamics with an Overset Grid Method

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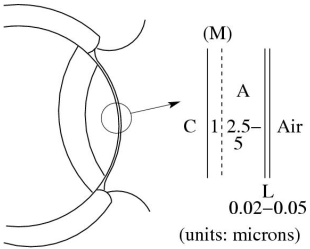
What is Human Tear Film?

A multilayer structure playing a vital role in health and function of the eye.



Typical thickness of each layer in microns.

Cornea: Modeled as a flat wall.

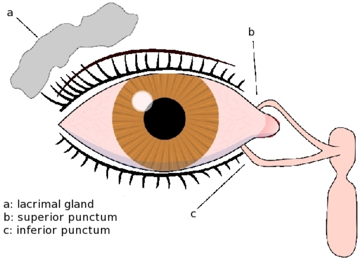
Layer 1: (M) is a mucus layer.

Layer 2: A is the aqueous layer, primarily water (about 98%).

Layer 3: L is the lipid layer, polar surfactants at the A/L interface.

Overview of the Dynamics

Tear film supply and drainage



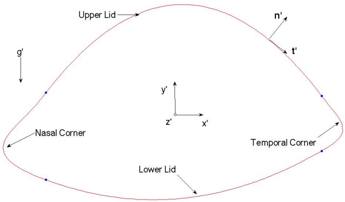
Lacrimal gland:

The lacrimal gland supplies new tear fluid.

Punctal drainage:

Removes excess fluid starting at the halfway open position of the lids.

Domain Ω



Boundary $\partial\Omega$

Upper/Lower Lid:

Polynomials fit to measure lid data.*

Temporal/Nasal Corner:

Polynomials constructed to create a smooth boundary.

*MATLAB program created by Xiaolin Yang.

Characteristic length scales

For x' and y' direction:

$L' = 5mm$, half width of cornea.

For z' direction:

$d' = 5\mu$m, thickness of film.

The ratio of length scales $\epsilon = d'/L' \approx 10^{-3} \Rightarrow$ lubrication.

Evolution Equation

The evolution of the free surface is given by

$$
\begin{aligned}
h_t + \nabla \cdot \left[-\frac{h^3}{12} \nabla (p + Gy) \right] &= 0, \\
p + S\Delta h &= 0,
\end{aligned}
$$

with surface tension $S \approx 1.0 \times 10^{-5}$ and gravity $G \approx 2.5 \times 10^{-3}$.

Boundary conditions

Thickness: $h|_{\partial\Omega} = h_0$

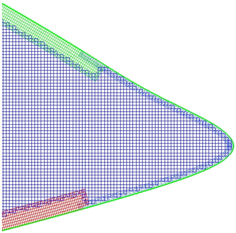
Flux: $\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = Q(s)$, s denotes arc length

Initial condition

$h(\mathbf{x}, 0) = (h_0 - 1)\, e^{-\min(\mathrm{dist}(\mathbf{x}, \partial\Omega))/x_0} + 1$, $x_0 = 0.1$ and $h_0 = 13$

Numerical Method

Overset grid: Generated in Overture.



Temporal corner.

Boundary curves:

Defined by NURBS mapping.

Grid for upper/lower lid:

Defined by a normal mapping.

Grid for temporal/nasal corner:

Defined by a normal mapping.

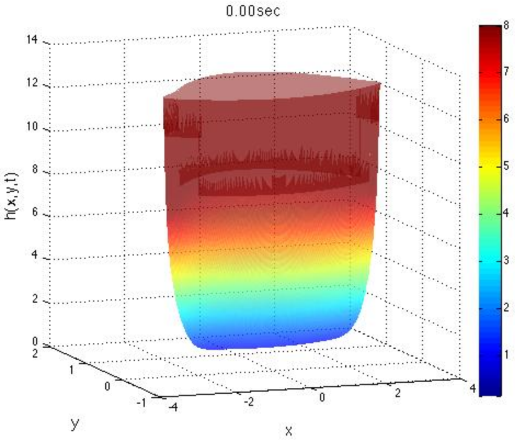
Method of lines

Spatial discretization: Second-order curvilinear finite differences generated in Overture.

Time discretization: Variable stepsize (fixed leading coefficient) second-order backward differentiation formula.

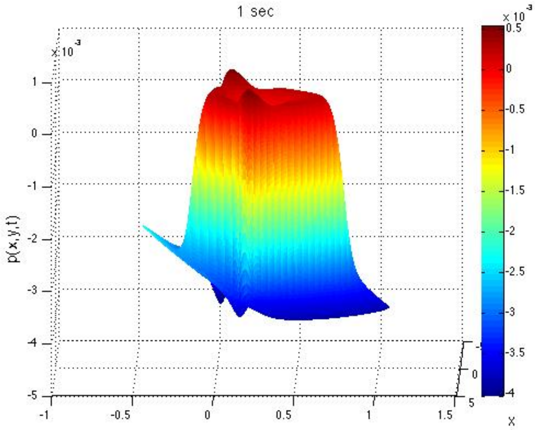
No Flux: Tear Film Thickness

Evolution of the tear film thickness



No Flux: What is Driving the Motion?

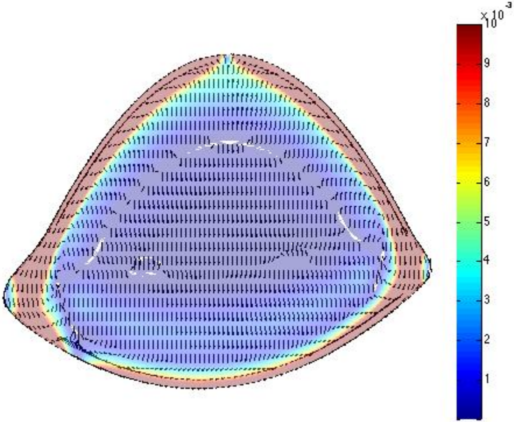
Pressure gradients



View from temporal corner of the pressure.

No Flux: What is Driving the Motion

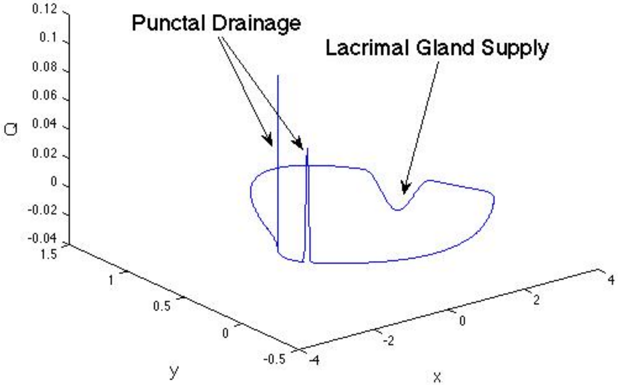
Gravity



Flux vector field: Gravity steers the fluid towards the lower lid.

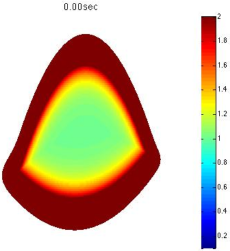
Flux: Boundary Condition

$$\mathbf{n} \cdot \mathbf{q} =$$

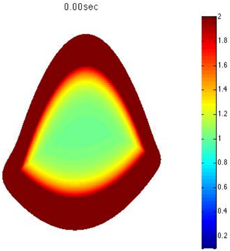


Flux: Tear Film Thickness

Evolution of the tear film thickness of interior



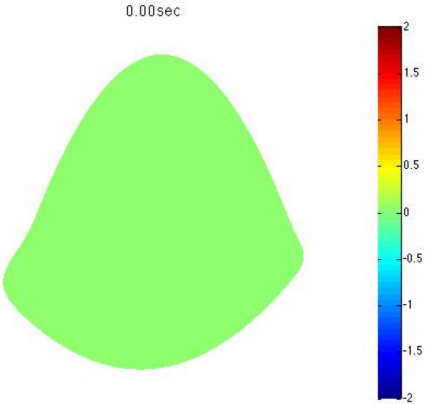
No Flux



Flux

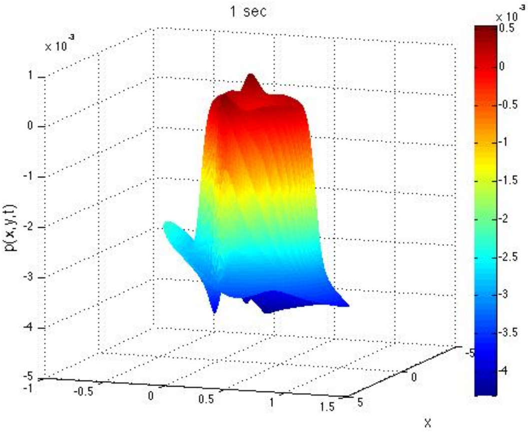
No Flux: Tear Film Thickness

Evolution of the tear film thickness of menisci



Flux: What is Driving the Motion?

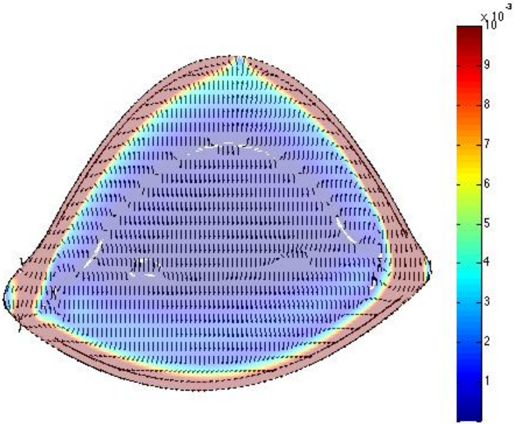
Pressure gradients



View from temporal corner of the pressure.

Flux: What is Driving the Motion

Gravity



Flux vector field: Outward fluxes in the nasal corner. Connectivity of the menisci.

Summary

Conclusions

Developed a numerical algorithm to study the tear film dynamics on the complex eye domain.

The curvature of the eye-shaped domain helps draw tear film into the corners.

Found evidence of hydraulic connectivity between the upper and lower menisci.

Gravity steers the tear fluid towards the lower lid.

Future work

Simulate on a moving domain.

Formulation

At leading order, on $0 < z < h(x, y, t)$ and $(x, y) \in \Omega$ we have

$$u_x + v_y + w_z = 0, \;\; u_{zz} - p_x = 0, \;\; v_{zz} - p_y - G = 0 \;\text{ and }\; p_z = 0.$$

Boundary conditions:

Eye surface: No-slip condition and impermeability
$u = 0$, $v = 0$, $w = 0$.

Free surface: Kinematic condition: $w = h_t + uh_x + vh_y$.
Normal stress condition: $-p = S\Delta h$.
Tangentially immobile surface: $u = 0$, $v = 0$.

The evolution of the free surface is

$$h_t + \nabla \cdot \left[\frac{h^3}{12} \nabla \left(-p - Gy \right) \right] = 0, \;\; p = -S\Delta h,$$

where $G = \frac{\rho g d'^2}{\mu U'_m}$, $S = \frac{\epsilon^3 \sigma}{\mu U'_m}$ and the flux is $Q = \frac{h^3}{12} \nabla \left(-p - Gy \right)$.