Single-Equation Models for the Tear Film in a Blink Cycle with Realistic Lid Motion

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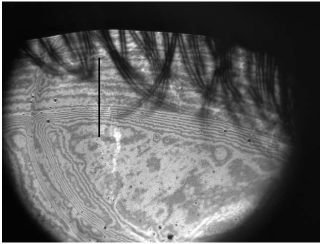
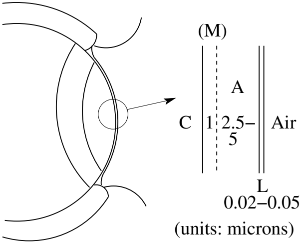
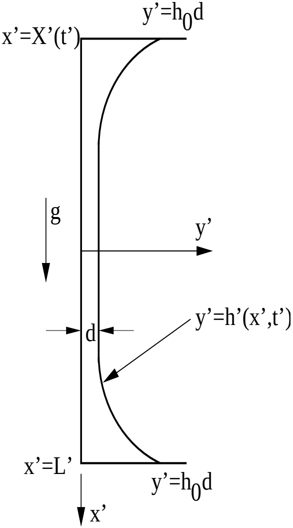
How do we simulate the dynamics of the tear film ?



Interference fringes.





$\epsilon = \frac{d}{L'}$ is small → Lubrication theory

Physical parameters: Braun et al.

Constants	Description
$L' = 5$ mm	half the width of the palpebral fissure (x direction)
$d = 5\ \mu$m	thickness of the tear film away from ends
$\epsilon = \frac{d}{L'} \approx 10^{-3}$	small parameter for lubrication theory
$U_m = 10$–30 cm/s	maximum speed across the film
$L'/U_m = 0.05$ s	time scale for real blink speeds
$\sigma_0 = 45$ mN/m	surface tension
$\mu = 10^{-3}$ Pa·s	viscosity
$\rho = 10^3$ kg/m^3	density

- Inside the film
  - Viscous incompressible parallel flow inside the film.
  - Inertial terms and gravity are neglected.
- At the impermeable wall $y = 0$
  - $v = 0, \quad u = \beta u_y;$
- At the free surface $y = h(x,t)$
  - Simplified stress conditions

$$p = -Sh_{xx}, \quad S = \frac{\epsilon^3 \sigma}{\mu U_m}, \quad u^{(s)} = X_t \frac{1-x}{1-X}$$

  - Kinematic condition

$$h_t + q_x = 0 \text{ on } X(t) \le x \le 1,$$

where

$$q = \int_0^h u(x,y,t)dy$$

- The uniform stretching limit (USL).

$$q(x,t) = \frac{h^3}{12}\left(1 + \frac{3\beta}{h+\beta}\right)(Sh_{xxx}) + X_t \frac{1-x}{1-X}\frac{h}{2}\left(1 + \frac{\beta}{h+\beta}\right)$$

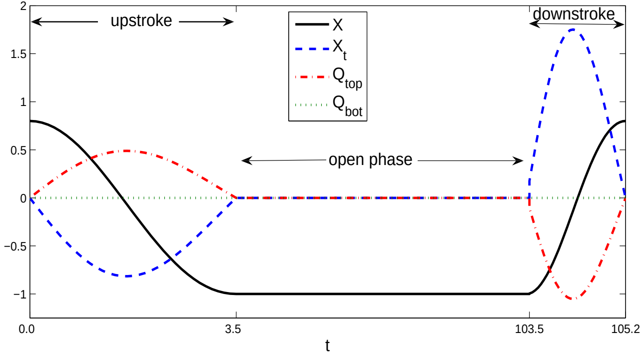
Boundary conditions

$$h(X(t),t) = h(1,t) = h_0 \quad q(X(t),t) = X_t h_0 + Q_{top} \quad q(1,t) = -Q_{bot}.$$

Initial condition

Polynomial function

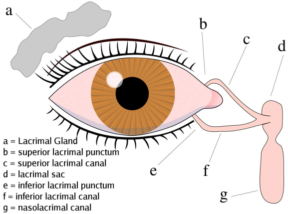
We depart from Berke and Mueller (98) ⇒ Heryudono et al (07)



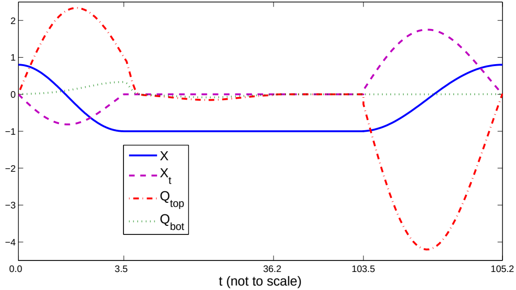
Flux proportional to lid motion (FPLM) (Jones et al (05))

$$Q_{top} = -X_t h_e, \quad Q_{bot} = 0$$

Add in lacrimal gland supply and punctal drainage approximated by Gaussians.



Picture above is taken from the Wikipedia commons



We transform the PDE into a fixed domain $\xi \in [-1, 1]$ via

$$\xi = 1 - 2\frac{1-x}{1-X(t)}.$$

The equations become

$$H_t = \frac{1-\xi}{L-X} X_t H_\xi - \left(\frac{2}{L-X}\right) Q_\xi$$

$$Q = X_t \frac{1-\xi}{2}\frac{H}{2}\left(1 + \frac{\beta}{H+\beta}\right) + \frac{H^3}{12}\left(1 + \frac{3\beta}{H+\beta}\right)\left[S\left(\frac{2}{1-X}\right)^3 H_{\xi\xi\xi}\right]$$

$$H(\pm 1, t) = h_0, \; Q(-1, t) = X_t h_0 + Q_{top}, \; Q(1, t) = -Q_{bot}, \quad (BCs)$$

$$H(\xi, 0) = h_m + (h_0 - h_m)\xi^m \quad (IC).$$

$\Rightarrow$ Spectral discretization in space and standard ODE in time.

- Use spectral collocation method.
  - Map Chebyshev points technique Kosloff & Tal-Ezer (93)
  - Use mapping parameter by Don & Solomonoff (97) to reduce roundoff errors near end points.
- Imposing boundary conditions.
  - Set Q

$$Q = X_t \frac{1-\xi}{2} \frac{H}{2} \left(1 + \frac{\beta}{H+\beta}\right) + \frac{H^3}{12} \left(1 + \frac{3\beta}{H+\beta}\right) \left[S \left(\frac{2}{1-X}\right)^3 H_{\xi\xi\xi} \right]$$

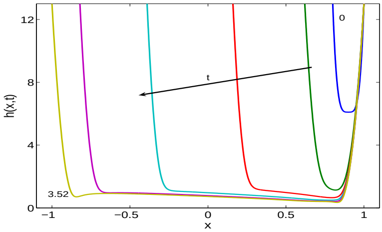
  - When computing Q_ξ, overwrite its end values with $Q(-1,t) = X_t h_0 + Q_{top}, \quad Q(1,t) = -Q_{bot}$.
- Solve the initial value problem at inner nodes.

$$H_t = \frac{1-\xi}{1-X} X_t H_\xi - \left(\frac{2}{1-X}\right) Q_\xi$$

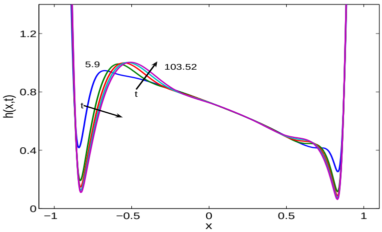
with ode solver ode15s.

Parameters $N = 351, \lambda = 0.1, \beta = 10^{-2}, S = 2 \times 10^{-5}, h_0 = 13, h_e = 0.6$, and initial volume $V_0 = 2.576$. Our simulation is done in MATLAB with ode15s as ODE solver.

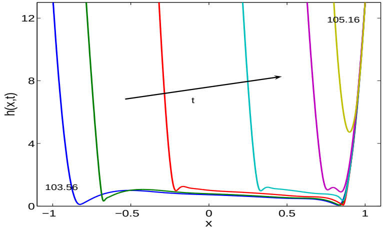
Opening phase



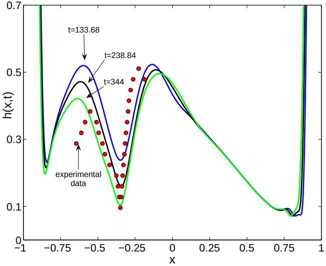
fully open (zoom)



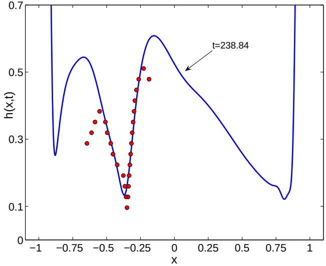
The closing phase

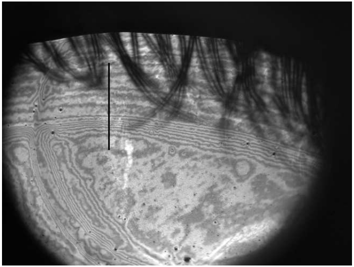


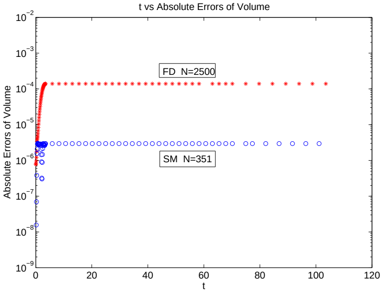
$S = 2 \times 10^{-5}$



$S = 8 \times 10^{-6}$







Conservation of volume

SUMMARY

- 1-D simulation of the tear film in a blink cycle.
- Good fit with experimental data for partial blink simulation with FPLM+ type fluxes.
- Use spectral methods for getting higher accuracy solutions.
- To appear in IMA Journal of Mathematical Medicine and Biology.