Single-Equation Models for the Tear Film in a Blink Cycle with Realistic Lid Motion

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How do we simulate the dynamics of the tear film?

Interference fringes.

\[
\begin{align*}
\epsilon &= \frac{d}{L} \text{ is small} 
\rightarrow \text{Lubrication theory}
\end{align*}
\]
Physical parameters: Braun et al.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Description</th>
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<tbody>
<tr>
<td>$L' = 5 \text{ mm}$</td>
<td>half the width of the palpebral fissure ($x$ direction)</td>
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<tr>
<td>$d = 5 \mu m$</td>
<td>thickness of the tear film away from ends</td>
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<td>$\epsilon = \frac{d}{L'} \approx 10^{-3}$</td>
<td>small parameter for lubrication theory</td>
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<td>$U_m = 10-30 \text{ cm/s}$</td>
<td>maximum speed across the film</td>
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<td>$L' / U_m = 0.05 \text{ s}$</td>
<td>time scale for real blink speeds</td>
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<td>$\sigma_0 = 45 \text{ mN/m}$</td>
<td>surface tension</td>
</tr>
<tr>
<td>$\mu = 10^{-3} \text{ Pa s}$</td>
<td>viscosity</td>
</tr>
<tr>
<td>$\rho = 10^3 \text{ kg/m}^3$</td>
<td>density</td>
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- **Inside the film**
  - Viscous incompressible parallel flow inside the film.
  - Inertial terms and gravity are neglected.

- **At the impermeable wall $y = 0$**
  - $\nu = 0, \quad u = \beta u_y$;

- **At the free surface $y = h(x, t)$**
  - Simplified stress conditions
    \[
    p = -Sh_{xx}, \quad S = \frac{\epsilon^3 \sigma}{\mu U_m}, \quad u^{(s)} = X_t \frac{1-x}{1-X}
    \]

- **Kinematic condition**
Problem Free surface evolution

\[ h_t + q_x = 0 \text{ on } X(t) \leq x \leq 1, \]

where

\[ q = \int_0^h u(x, y, t)dy \]

- The uniform stretching limit (USL).

\[ q(x, t) = \frac{h^3}{12} \left( 1 + \frac{3\beta}{h + \beta} \right) (Sh_{xxx}) + X_t \frac{1 - x}{1 - X} \frac{h}{2} \left( 1 + \frac{\beta}{h + \beta} \right) \]

Boundary conditions

\[ h(X(t), t) = h(1, t) = h_0 \quad q(X(t), t) = X_t h_0 + Q_{top} \quad q(1, t) = -Q_{bot}. \]

Initial condition

Polynomial function

Heryudono et al (1 Math Sciences, U of Delaware and 2 College of Optometry, Ohio State U) Single-Equation models for the Tear Film
We depart from Berke and Mueller (98) ⇒ Heryudono et al (07)

Flux proportional to lid motion (FPLM) (Jones et al (05))

\[ Q_{top} = -X_t h_e, \quad Q_{bot} = 0 \]
Add in lacrimal gland supply and punctal drainage approximated by Gaussians.
We transform the PDE into a fixed domain $\xi \in [-1, 1]$ via

$$\xi = 1 - 2 \frac{1 - x}{1 - X(t)}.$$

The equations become

$$H_t = \frac{1 - \xi}{L - X} X_t H_\xi - \left( \frac{2}{L - X} \right) Q_\xi$$

$$Q = X_t \frac{1 - \xi}{2} H \left( 1 + \frac{\beta}{H + \beta} \right) + \frac{H^3}{12} \left( 1 + \frac{3\beta}{H + \beta} \right) \left[ S \left( \frac{2}{1 - X} \right)^3 H_{\xi\xi\xi} \right]$$

$H(\pm1, t) = h_0$, $Q(-1, t) = X_t h_0 + Q_{top}$, $Q(1, t) = -Q_{bot}$, \hspace{1cm} (BCs)

$H(\xi, 0) = h_m + (h_0 - h_m)\xi^m$ \hspace{1cm} (IC).

$\Rightarrow$ Spectral discretization in space and standard ODE in time.
Use spectral collocation method.

- Map Chebyshev points technique Kosloff & Tal-Ezer (93)
- Use mapping parameter by Don & Solomonoff (97) to reduce roundoff errors near end points.

Imposing boundary conditions.

- Set \( Q \)

\[
Q = X_t \frac{1 - \xi}{2} \frac{H}{2} \left( 1 + \frac{\beta}{H + \beta} \right) + \frac{H^3}{12} \left( 1 + \frac{3\beta}{H + \beta} \right) \left[ S \left( \frac{2}{1 - X} \right)^3 H_{\xi \xi \xi} \right]
\]

- When computing \( Q_\xi \), overwrite its end values with \( Q(-1, t) = X_t h_0 + Q_{top}, \quad Q(1, t) = -Q_{bot} \).

Solve the initial value problem at inner nodes.

\[
H_t = \frac{1 - \xi}{1 - X} X_t H_\xi - \left( \frac{2}{1 - X} \right) Q_\xi
\]

with ode solver ode15s.
Parameters $N = 351, \lambda = 0.1, \beta = 10^{-2}, S = 2 \times 10^{-5}, h_0 = 13, h_e = 0.6$, and initial volume $V_0 = 2.576$. Our simulation is done in MATLAB with ode15s as ODE solver.
Partial blinks results: FPLM + Gaussians

Numerical results

Experimental data

Conservation of volume

FD $N=2500$

SM $N=351$

Single-Equation models for the Tear Film

Heryudono et al
SUMMARY

• 1-D simulation of the tear film in a blink cycle.
• Good fit with experimental data for partial blink simulation with FPLM+ type fluxes.
• Use spectral methods for getting higher accuracy solutions.
• To appear in IMA Journal of Mathematical Medicine and Biology.