**Homework 8. (Solutions to selected problems)**
Math 353 Section 12, Fall 2008, University of Delaware

1. **Solution** (a) Euler method formula is

$$y_0 = y(1) = 2; \quad y_{k+1} = y_k + h\left(1 + \frac{y_k}{t_k}\right) = h + y_k\left(1 + \frac{h}{t_k}\right).$$

For the step size $h = 0.25$, the nodes are $t_0 = 1,\ t_1 = 1.25,\ t_2 = 1.5,\ t_3 = 1.75,\ t_4 = 2$ and the calculations are

$$y_1 = 2.75, \quad y_2 = 3.55, \quad y_3 = 4.3917, \quad y_4 = 5.2690.$$

Error Table

| $t_k$ | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|
| $\lvert y(t_k) - y_k\rvert$ | 0 | 0.0289 | 0.0582 | 0.0877 | 0.1172 |

(b) First, we find Lipschitz constant $L$, on $[1, 2]$,

$$|f(t, y_1) - f(t, y_2)| = \frac{1}{t}|y_1 - y_2| \le |y_1 - y_2|, \text{ as } t \in [1,2] \Rightarrow L = 1.$$

Now we find

$$M = max_{1\le c\le 2}|y''(c)| = max_{1\le c\le 2}\left|\frac{1}{c}\right| = 1$$

Therefore,

$$g_k \le \frac{Mh}{2L}|(e^{L(t_k-a)} - 1)| \le \frac{h}{2}(e-1) \approx 0.8591h.$$

(c) Explicit Trapezoid method formula is

$$y_0 = 2, \quad y_{k+1} = y_k + \frac{h}{2}\left(2 + \frac{y_k}{t_k} + \frac{y_k t_k + h(y_k + t_k)}{t_k(t_k + h)}\right).$$

For the step size $h = 0.25$, the nodes are $t_0 = 1,\ t_1 = 1.25,\ t_2 = 1.5,\ t_3 = 1.75,\ t_4 = 2$ and the calculations are

$$y_1 = 2.775, \quad y_2 = 3.6008, \quad y_3 = 4.46882, \quad y_4 = 5.37285.$$

Error Table

| $t_k$ | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|
| $\lvert y(t_k) - y_k\rvert$ | 0 | 0.00392 | 0.0073 | 0.0104 | 0.01343 |

2. Exercise 6.2.1.

3. Exercise 6.2.4 (e). **Solution.** First we find its exact solution

$$y' = \frac{dy}{dt} = \frac{1}{y^2} \Rightarrow \int y^2\, dy = \int 1\, dt \Rightarrow y(t) = \sqrt[3]{3t + C} \text{ and } y(0) = 1 \Rightarrow y(t) = \sqrt[3]{3t+1}.$$

Now we find

$$\frac{d}{dt}\left(\frac{1}{y^2}\right) = -\frac{2}{y^3}y' = -\frac{2}{y^5}.$$

Therefore, Taylor method of order 2 formula is

$$y_0 = 1, \quad y_{k+1} = y_k + \frac{h}{y_k^2} - \frac{h^2}{2}2y_k^5 = \frac{y_k^6 + hy_k^3 - h^2}{y_k^5}.$$

For the step size $h = 0.25$, the nodes are $t_0 = 0,\ t_1 = 0.25,\ t_2 = 0.5,\ t_3 = 0.75,\ t_4 = 1$ and the calculations are

$$y_1 = 1.1875, \quad y_2 = 1.3383, \quad y_3 = 1.4633, \quad y_4 = 1.5708.$$

Error at $t = 1$ is $|y(1) - y_4| = 0.01662$.

4. Exercise 6.4.1 (b). **Solution.** First we find its exact solution

$$y' = \frac{dy}{dt} = t^2 y \Rightarrow \int \frac{1}{y}\, dy = \int t^2\, dt \Rightarrow \ln|y| = \frac{t^3}{3} + C \text{ and } y(0) = 1 \Rightarrow y(t) = e^{t^3/3}.$$

Midpoint method formula is

$$y_0 = 1, \quad y_{k+1} = y_k + h\left(t_k + \frac{h}{2}\right)^2\left(y_k + \frac{ht_k^2 y_k}{2}\right) = \frac{y_k}{8}(8 + h(2t_k + h)^2(2 + ht_k^2)).$$

For the step size $h = 0.25$, the nodes are $t_0 = 0,\ t_1 = 0.25,\ t_2 = 0.5,\ t_3 = 0.75,\ t_4 = 1$ and the calculations are

$$y_1 = 1.0039062, \quad y_2 = 1.0395, \quad y_3 = 1.1442, \quad y_4 = 1.3786.$$

Error at $t = 1$ is $|y(1) - y_4| = 0.0171$.

5. Exercise 6.4.3 (b). For the step size $h = 0.25$, the nodes are $t_0 = 0,\ t_1 = 0.25,\ t_2 = 0.5,\ t_3 = 0.75,\ t_4 = 1$ and the calculations are

$$y_1 = 1.0052, \quad y_2 = 1.0425, \quad y_3 = 1.1510, \quad y_4 = 1.3956.$$

and error table is

| $t_k$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $\lvert y(t_k) - y_k \rvert$ | 0 | 0.000021 | 0.000046 | 0.000007 | 0.000012 |