Solving Non-Linear Equation.
Math 353, Fall 2008, University of Delaware

Definition. If $f(c_0) = 0$, then we say number c_0 is a solution (root) of the equation $f(x) = 0$.

Bisection Method

Let f is continuous on the given initial interval $[a, b]$ and $f(a)f(b) < 0$.

1. Define $c := \frac{a+b}{2}$.
2. If $b - c \leq \varepsilon$, then accept root:=c, and stop.
3. If $f(b)f(c) < 0$, then $a := c$; else $b := c$.
4. Return to step 1.

Regula-Falsi Method

Let f is continuous on the given initial interval $[a, b]$ and $f(a)f(b) < 0$.

1. Define
$$c := \frac{bf(a) - af(b)}{f(a) - f(b)}.$$
2. If $c - a \leq \varepsilon$, then accept root:=c, and stop.
3. If $f(c) = 0$, then accept root:=c, and stop.
4. If $f(b)f(c) < 0$, then $a := c$; else $b := c$.
5. Return to step 1.

Definition. A fixed point of a function f is a number p such that $f(p) = p$.

Fixed-Point Method

1. x_0 is an initial guess.
2. for $i = 1, 2, ...$
$$x_i = f(x_{i-1}).$$

Newton-Raphson Method

1. x_0 is an initial guess.
2. for $i = 1, 2, ...$
$$x_i = x_{i-1} - \frac{mf(x_{i-1})}{f'(x_{i-1})}.$$

where, $m = 1$ for a simple root and $m > 1$ is an order of a multiple root.

Secant Method

1. x_0 and x_1 are initial guesses.
2. for $i = 1, 2, ...$
$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}.$$

Definition. Assume x_A is an approximation to x. Then

$$E = |x - x_A| \text{ is called the absolute error}$$

and

$$R = \frac{|x - x_A|}{|x|} \text{ is called the relative error,}$$

provided that $x \neq 0$.

Definition. Let f has a root at c and x_c is an approximation to c. Then, for the root-finding problem, the forward error is $|c - x_c|$ and the backward error is $|f(x_c)|$.

Definition. Let f has a root at c and x_c is an approximation to c produced by a root finding algorithm. Then, we define its

$$\text{error magnification factor} = \frac{\text{relative forward error}}{\text{relative backward error}}.$$

Definition. Assume that $\{c_n\}_{n=0}^{\infty}$ converges to c and set $e_n = |c - c_n|$ for $n \geq 0$. If two positive constant S and k exist, and

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^k} = S,$$

then the sequence is said to converge to c with order of convergence k and with rate S.
If $k = 1$, the convergence is called linear.
If $k = 2$, the convergence is called quadratic.

Definition. A root-finding method is called locally convergent if the method converges to a root for initial guesses sufficiently close to the root.

Comparison of the methods

Method	Root	Convergence	Order	Rate
Bisection	Simple/Multiple	Global	Linear	$\approx \frac{1}{2}$
Regula Falsi	Simple/Multiple	Global	Linear	—
Fixed Point	—	Local	Linear	$\approx \lvert f'(c)\rvert$
Secant	Simple	Local	$\frac{1+\sqrt{5}}{2} \approx 1.618$	$\approx \left\lvert \frac{f''(c)}{2f'(c)} \right\rvert^{0.618}$
Secant	Multiple	Local	Linear	—
Newton-Raphson	Simple	Local	Quadratic	$\approx \left\lvert \frac{f''(c)}{2f'(c)} \right\rvert$
Newton-Raphson	Multiple	Local	Linear	$\approx \frac{m-1}{m}$
Modified Newton	Multiple	Local	Quadratic	—

MatLab command:
>>fzero('f(x)',[a,b]) - Finds a root of $f(x)$ in the interval $[a, b]$.
>>fzero('f(x)',a) - Finds a root of $f(x)$ near $x = a$ point.
It uses a hybrid method (Brent's Method).