Numerical Integration.
Math 353 Section 12, Fall 2008, University of Delaware

Definition. Let $a \le x_0 < x_1 < ... < x_n \le b$. A formula of the form

$$Q[f] = \sum_{k=0}^{m} w_k f(x_k)$$

such that

$$\int_a^b f(x)\,dx = Q[f] + E[f], \quad (\int_a^b f(x)\,dx \approx Q[f])$$

is called an quadrature formula (numerical integration).
$E[f]$ is called the truncation error of the quadrature.
$x_0, x_1, ..., x_m$ are called the quadrature nodes and $w_0, w_1, ..., w_m$ are called the weights.
Definition. The degree of precision of a quadrature formula, n, is the positive integer number such that $E[P_k(x)] =$ for all polynomial of degree $k \le n$, but $E[P_{n+1}(x)] \ne 0$ for some polynomial of degree $n+1$.
If quadrature formula derived by polynomial interpolation of f on $[a,b]$ using the equally-spaced nodes with $x_0 = a$ and $x_n = b$, then resulting formula is called a closed Newton-Cotes quadrature.
Let $x_k = x_0 + kh, \;\; k = 1, 2, ...$ and $f_k = f(x_k), \;\; k = 0, 1, ...$
The first four closed Newton-Cotes quadratures are

1. Trapezoidal Rule ($f \in C^2[a,b]$)

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} f''(c), \quad n = 1$$

2. Simpson's Rule ($f \in C^4[a,b]$)

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(c), \quad n = 3$$

3. Simpson's $\frac{3}{8}$ Rule ($f \in C^4[a,b]$)

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(c), \quad n = 3$$

4. Boole's Rule ($f \in C^6[a,b]$)

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945} f^{(6)}(c), \quad n = 5.$$

Composite Trapezoid Rule Let $x_0 = a, \;\; x_k = a + kh$, and $f_k = f(x_k), \;\; k = 0, 1, ..., m$, where $h = \frac{b-a}{m}$. Then

$$\int_a^b f(x)\,dx = \left[\frac{h}{2}(f_0 + f_m) + h\sum_{k=1}^{m-1} f_k\right] - \frac{(b-a)h^2}{12} f''(c),$$

where c in (a, b).
Composite Simpson's Rule Let $x_0 = a$, $x_k = a + kh$, and $f_k = f(x_k)$, $k = 0, 1, ..., 2m$, where $h = \frac{b-a}{2m}$. Then

$$\int_a^b f(x)\,dx = \left[\frac{h}{3}(f_0 + f_{2m}) + \frac{2h}{3}\sum_{k=1}^{m-1} f_{2k} + \frac{4h}{3}\sum_{k=1}^{m} f_{2k-1}\right] - \frac{(b-a)h^4}{180} f^{(4)}(c),$$

where c in (a, b).
Composite Simpson's $\frac{3}{8}$ Rule Let $x_0 = a$, $x_k = a + kh$, and $f_k = f(x_k)$, $k = 0, 1, ..., 3m$, where $h = \frac{b-a}{3m}$. Then

$$\int_a^b f(x)\,dx = \left[\frac{3h}{8}\sum_{k=1}^{m}(f_{3k-3} + 3f_{3k-2} + 3f_{3k-1} + f_{3k})\right] - \frac{(b-a)h^4}{80} f^{(4)}(c),$$

where c in (a, b).
Composite Midpoint Rule (open formula) Let $x_0 = a$, $x_k = a + kh$, and $f_k = f(x_k)$, $k = 0, 1, ..., 2m$, where $h = \frac{b-a}{2m}$. Then

$$\int_a^b f(x)\,dx = \left[2h\sum_{k=1}^{m} f_{2k-1}\right] + \frac{(b-a)h^2}{6} f''(c),$$

where c in (a, b).
Romberg Integration
Romberg integration uses the Composite Trapezoid rule to give preliminary approximations and then applies the Richardson's extrapolation to improve the approximations.

j	$h_j = (b-a)/2^{j-1}$	R_{j1}- Trapezoid $O(h^2)$	R_{j2}- Simpson's $O(h^4)$	R_{j3}- Boole's $O(h^6)$	$\cdots$
1	$h_1 = (b-a)$	$R_{11} = (h_1/2)(f(a) + f(b))$	—	—	$\cdots$
2	$h_2 = (b-a)/2$	$R_{21} = R_{11}/2 + h_2 f_1$	$R_{22} = (4R_{21} - R_{11})/3$	—	
3	$h_3 = (b-a)/4$	$R_{31} = R_{22}/2 + h_3(f_1 + f_3)$	$R_{32} = (4R_{31} - R_{21})/3$	$R_{33} = (16R_{32} - R_{22})/15$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

general recursive formula is (for k=1)

$$R_{jk} = R_{j1} = \frac{1}{2}R_{j-11} + h_j \sum_{i=1}^{2^{j-2}} f(a + (2i-1)h_j)$$

and for $k > 2$

$$R_{jk} = \frac{4^{k-1}R_{jk-1} - R_{j-1k-1}}{4^{k-1} - 1}.$$

Legendre Polynomials. For positive integer n:

$$p_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}((x^2 - 1)^k), \quad k = 0, 1, ..., n$$

is orthogonal set of polynomials on $[-1, 1]$.
Gauss-Legendre quadrature.

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{m} w_k f(x_k),$$

where $x_1, ..., x_m$ are roots of m th Legendre polynomial and

$$w_k = \int_{-1}^{1} L_k(x)\, dx, \quad k = 1, 2, ...m.$$

Here, $L_k(x)$ is Lagrange polynomial;

$$L_k(x) = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_m)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_m)}.$$

On a general interval $[a, b]$:

$$\int_a^b f(x)\, dx = \int_{-1}^{1} f\left(\frac{(b - a)t + b + a}{2}\right) \frac{b - a}{2}\, dt.$$