

Numerical Differentiation.
Math 353 Section 12, Fall 2008, University of Delaware

Finite Difference Formulas.
Formulas of order 1 ($O(h)$).

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad - \quad \text{Forward difference} \quad h_{optimal} \approx 2\sqrt{\frac{\varepsilon_{mach}}{|f''(x)|}}$$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \quad - \quad \text{Backward difference}$$

Centered-Difference formulas of order 2 ($O(h^2)$).

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \quad h_{optimal} \approx \left(\frac{3\varepsilon_{mach}}{|f'''(x)|}\right)^{1/3}$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \quad h_{optimal} \approx \left(\frac{48\varepsilon_{mach}}{|f^{(iv)}(x)|}\right)^{1/4}$$

$$f'''(x) \approx \frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$$

Forward and Backward formulas order of 2 ($O(h^2)$).

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \quad - \quad \text{Forward difference}$$

$$f'(x) \approx \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} \quad - \quad \text{Backward difference}$$

$$f''(x) \approx \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2} \quad - \quad \text{Forward difference}$$

$$f''(x) \approx \frac{-2f(x) + 5f(x-h) - 4f(x-2h) + f(x-3h)}{h^2} \quad - \quad \text{Backward difference}$$

Here, $0 < h \ll 1$ is and f is defined on $[x - 3h, x + 3h]$.

The truncation error is defined

$$E_{trunc} = |f'(x)_{true} - f'(x)_{formula}|$$

The total error of the difference formula is

$$E_{total} = |f'(x)_{true} - f'(x)_{machine}| \approx E_{trunc} + E_{rounding}.$$

Richardson Extrapolation.

Let we have an order n formula $D(h)$ for approximating a given quantity P ;

$$P = D(h) + O(h^n)$$

Then

$$P = \frac{2^n D(h/2) - D(h)}{2^n - 1} + O(h^{n+1}).$$