Midterm Exam 2. (Solutions)
Math 353 Sections 12, Fall 2008. University of Delaware

1. A natural cubic spline $S(x)$ is defined by

$$S(x) = \begin{cases} S_1(x) = 1 + b_1(x-1) - d_1(x-1)^3 & \text{on } [1,2] \\ S_2(x) = 2 + b_2(x-2) - \frac{3}{4}(x-2)^2 + d_2(x-2)^3 & \text{on } [2,3]. \end{cases}$$

If $S(x)$ interpolates the data $(1,1),\ (2,1),\ (3,0)$, then find $b_1,\ b_2,\ d_1,\ d_2$.
Solution. Differences are

$$\delta_1 = \delta_2 = 1, \quad \triangle_1 = 0, \quad \triangle_2 = -1.$$

The the tridiagonal matrix equation for $c = (0, -3/4, c_3)^\top$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3/4 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$$

The solution is $c = (0, -3/4, 0)^\top$. Now we find

$$-d_1 = \frac{c_2 - c_1}{3\delta_1} = -\frac{1}{4}, \quad d_2 = \frac{c_3 - c_2}{3\delta_2} = \frac{1}{4}$$

and

$$b_1 = \frac{\triangle_1}{\delta_1} - \frac{\delta_1}{3}(2c_1 + c_2) = \frac{1}{4}, \quad b_2 = \frac{\triangle_2}{\delta_2} - \frac{\delta_1}{3}(2c_2 + c_3) = -\frac{1}{2}.$$

2. Construct

   (a) Lagrange

   (b) Newton's

   interpolating polynomials for the following function, and find an error bound for the approximation at the given point.

$$f(x) = \cos\left(\frac{\pi}{2}x\right), \quad x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3 \text{ and } x = 1.5$$

Solution. Computing the function values at the nodes, we get the data we have to interpolate

x	0	1	2	3
y	1	0	-1	0

. Lagrange form is

$$L(x) = 1\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} - 1\frac{x(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{x^3 - 3x^2 - x + 3}{3}.$$

The divided difference triangle is

0	1	-1	0	1/3
1	0	-1	1	
2	-1	1		
3	0			

thus, Newton's polynomial is $P(x) = 1 - x + \frac{1}{3}x(x-1)(x-2)$.

$$|f(1.5) - P(1.5)| \le \frac{(1.5-0)(1.5-1)(1.5-2)(1.5-3)}{4!}|f''''(c)| \le 0.0234\left(\frac{\pi}{2}\right)^4 \approx 0.1427$$

3. For the given set of data find the best least squares curve:

   (a) $y = p_1 + p_2 x$

   (b) $y = p_1 e^{p_2 x}$ (use the linearization)

x_k	1	2	3
y_k	1	2	5

Solution. (a) Substituting the data points into $y = p_1 + p_2 x$, we get

$1 = p_1 + p_2$

$2 = p_1 + 2p_2$ or, in matrix form,

$4 = p_1 + 3p_2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

The normal equations are

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \end{pmatrix}$$

Solving for the p_1 and p_2 results in the best line $y = -1.3333 + 2x$.

(b) Substituting the data into the linearized model $\ln y = \ln p_1 + p_2 x$ and we get

$\ln(1) = \ln(p_1) + p_2$

$\ln(2) = \ln(p_1) + 2p_2$ or, in matrix form,

$\ln(5) = \ln(p_1) + 3p_2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \ln p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(2) \\ \ln(5) \end{pmatrix}.$$

The normal equations are

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} \ln p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2.3026 \\ 6.2146 \end{pmatrix}$$

and by solving it we have $p_1 = 0.4309$ and $p_2 = 0.8047$.

4. Approximate the following integral using

   (a) Composite Trapezoid Rule with m=2

   (b) Composite Simpson's Rule with m=2

   (c) Composite Midpoint Rule with m=2

   (d) Romberg Integration, R_{22}

   (e) Gauss-Legendre quadrature with m=2

$$\int_{-1}^{1} \frac{x}{x+2}\, dx$$

Solution Let $f(x) = \frac{x}{x+2}$.

(a) $m = 2 \;\Rightarrow\; h = 1,\;\; x_0 = -1,\; x_1 = 0,\; x_2 = 1$, therefore

$$\int_{-1}^{1} \frac{x}{x+2}\, dx \approx \frac{h}{2}(f(-1) + f(1)) + hf(0) = -\frac{1}{3}$$

(b) $m = 2 \;\Rightarrow\; h = 1/2,\;\; x_0 = -1,\; x_1 = -0.5,\; x_2 = 0,\; x_3 = 0.5,\; x_4 = 1$, therefore

$$\int_{-1}^{1} \frac{x}{x+2}\, dx \approx \frac{h}{3}(f(-1) + f(1)) + \frac{4h}{3}(f(-0.5) + f(0.5)) + \frac{2h}{3}f(0) = -0.2$$

(c) $m = 2 \Rightarrow h = 1/2,\ x_0 = -1,\ x_1 = -0.5,\ x_2 = 0,\ x_3 = 0.5,\ x_4 = 1$, therefore

$$\int_{-1}^{1} \frac{x}{x+2}\, dx \approx 2h(f(-0.5) + f(0.5)) = -0.1333,$$

(d)

$$R_{11} = \frac{(1-(-1))}{2}(f(-1)+f(1)) = -\frac{2}{3},\ \ R_{21} = \frac{R_{11}}{2} + \frac{(1-(-1))}{2}f(0) = -\frac{1}{3},$$

therefore

$$R_{22} = \frac{4R_{21} - R_{11}}{3} = -\frac{2}{9}$$

(e) $m = 2 \Rightarrow x_1 = -1/\sqrt{3},\ x_2 = 1/\sqrt{3}$ and $w_1 = 1,\ w_2 = 1$, therefore

$$\int_{-1}^{1} \frac{x}{x+2}\, dx \approx f\Big(-\frac{1}{\sqrt{3}}\Big) + f\Big(\frac{1}{\sqrt{3}}\Big) = -0.1818$$

5. Given the initial value problem

$$y' = -t^2 y, \quad y(0) = 1, \quad 0 \le t \le 1$$

with exact solution $y(t) = e^{-t^3/3}$.

(a) Find the formula for Euler method. Use it with step size $h = 0.5$ to approximate the solution.

(b) Find a global error bound for Euler method.

Solution (a) Euler method formula is

$$y_0 = 1; \quad y_{k+1} = y_k + h(-t_k^2 y_k) = y_k(1 - ht_k^2).$$

For the step size $h = 0.5$, the calculations are

$$y_1 = 1(1-0) = 1,\ \ y_2 = 1(1 - 0.5 * 0.5^2) = 0.8750.$$

(b) First, we find Lipschitz constant L, on $[0,1]$,

$$|f(t,y_1) - f(t,y_2)| = t^2|y_1 - y_2| \le |y_1 - y_2|, \ \text{ as } \ t \in\ [0,1] \ \ \Rightarrow \ \ L = 1.$$

Now we find

$$M = max_{0\le c\le 1}|y''(c)| = max_{0\le c\le 1}|(c^4 - 2c)e^{-c^3/3}| \approx 1.035.$$

Therefore,

$$g_k \le \frac{Mh}{2L}|(e^{L(t_k - a)} - 1)| \le \frac{1.035h}{2}(e-1) \approx 0.8892h.$$

6. Given the initial value problem

$$y' = \frac{t-y}{2}, \quad y(0) = 1, \quad 0 \le t \le 2$$

(a) Use Taylor method of order 2 with $h = 1$ to approximate the solution.

(b) Use Midpoint method (RK2) with $h = 1$ to approximate the solution.

Solution. (a) First we find

$$f'(t,y) = \frac{d}{dt}(f(t,y)) = \frac{d}{dt}\left(\frac{t-y}{2}\right) = \frac{2-t+y}{4}.$$

Then Taylor method of order 2 formula is

$$y_0 = 1, \quad y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} f'(t_k, y_k) = \left(1 - \frac{h}{2} + \frac{h^2}{8}\right) y_k + \left(\frac{h}{2} - \frac{h^2}{8}\right) t_k + \frac{h^2}{4}.$$

For the step size $h = 1$, we find

$$y_{k+1} = \frac{5}{8} y_k + \frac{3}{8} t_k + \frac{1}{4} \quad \Rightarrow \quad y_1 = \frac{7}{8} = 0.875, \ y_2 = 1.1718.$$

(b) Midpoint method formula for the given problem is

$$y_0 = 1; \ \ y_{k+1} = y_k + \frac{(4h - h^2)(t_k - y_k) + 2h^2}{8}.$$

For the step size $h = 1$, we find

$$y_{k+1} = \frac{5y_k + 3t_k + 2}{8} \quad \Rightarrow \quad y_1 = \frac{7}{8} = 0.875, \ \ y_2 = 1.1718.$$