Midterm Exam 1. (Solutions)
Math 353 Sections 12, Fall 2008. University of Delaware

1. Solve the following system by the naive Gaussian Elimination. Show all your steps. (10pts)
$x_1 + 3x_3 = 3$
$3x_1 + x_2 + 6x_3 = 8$
$-5x_1 + 2x_2 - x_3 = 3$
Solution. Step 1. *Forward Elimination.*
Column 1:

$$\begin{pmatrix} 1 & 0 & 3 & |3 \\ 3 & 1 & 6 & |8 \\ -5 & 2 & -1 & |3 \end{pmatrix} \quad \frac{Eq2:\ Eq2 - 3Eq1}{Eq3:\ Eq3 + 5Eq1} \quad \begin{pmatrix} 1 & 0 & 3 & |3 \\ 0 & 1 & -3 & |-1 \\ 0 & 2 & 14 & |18 \end{pmatrix}$$

Column 2:

$$\begin{pmatrix} 1 & 0 & 3 & |3 \\ 0 & 1 & -3 & |-1 \\ 0 & 2 & 14 & |18 \end{pmatrix} \quad Eq3:\ Eq3 - 2Eq2 \quad \begin{pmatrix} 1 & 0 & 3 & |3 \\ 0 & 1 & -3 & |-1 \\ 0 & 0 & 20 & |20 \end{pmatrix}$$

Step 2. *Backward Substitution.*

$$20x_3 = 20 \Rightarrow x_3 = 1, \quad x_2 - 3x_3 = -1 \Rightarrow x_2 = 2, \quad x_1 + 3x_3 = 3 \Rightarrow x_1 = 0.$$

So, $x = (0, 2, 1)^\top$.

2. Consider the equation $e^x = x^2 - 3x + 2$.
   (a) Use the Intermediate Value Theorem to find an interval that contains a root of the equations. (3pts)
   (b) Apply two steps of the Bisection Method for the equation. (6pts)
   (c) Apply two steps of the Regula-Falsi Method for the equation. (6pts)

Solution. (a): Let $f(x) = e^x - x^2 + 3x - 2$. Then $f(0) = -1 < 0$ and $f(1) = e > 0$, thus $[a, b] = [0, 1]$.
(b):

iteration	a	$sign(f(a))$	c	$sign(f(c))$	b	$sign(f(b))$
0	0	-	0.5	+	1	+
1	0	-	0.25	-	0.5	+
2	0.25	-	0.375	+	0.5	+

(c):

iteration	a	$sign(f(a))$	c	$sign(f(c))$	b	$sign(f(b))$
0	0	-	0.2689	+	1	+
1	0	-	0.2578	+	0.2689	+
2	0	-	0.2575	+	0.2578	+

3. Let consider the equation

$$\frac{7}{x} + 8x^2 = 1.$$

It has unique solution $x = -1$.

   (a) Manipulate the equation to the Fixed-Point problem 2 different ways. (5pts)

(b) For each case determine whether Fixed-Point Iteration is convergent (at least one has to be convergent). (7pts)

(c) Find the forward and backward errors of the approximate root $x_A = -1.0234$. (3pts)

Solution. (a):

$$x = 8x^3 + 7 = g_1(x) \quad \text{and} \quad x = \frac{\sqrt[3]{x-7}}{2} = g_2(x)$$

(b):

$$g_1'(x) = 24x^2 \;\Rightarrow\; |g_1'(-1)| = 24 > 1 \;-\; \text{FPI is divergent}$$

$$g_2'(x) = \frac{1}{6\sqrt[3]{(x-7)^2}} \;\Rightarrow\; |g_2'(-1)| = \frac{1}{24} < 1 \;-\; \text{FPI is convergent}$$

(c): Let $x_{solution} = -1$ and

$$f(x) = \frac{7}{x} + 8x^2 - 1.$$

Then

$$\text{forward error} = |x_{solution} - x_A| = |-1 - (-1.0234)| = 0.0234,$$

$$\text{backward error} = |f(x_A)| = |f(-1.0234)| = 0.5388.$$

4. Let consider $f(x) = x^2 e^x = 0$.

(a) Determine the multiplicity of the root $x = 0$. (3pts)

(b) Find the Newton's Method formula and rate of convergence. (6pts)

(c) Apply two steps of the modified Newton's Method with initial guess $x_0 = 0.1$ for the equation. (6pts)

Solution. (a):

$$f'(x) = e^x x(x+2), \quad f''(x) = e^x(x^2 + 3x + 2) \;\Rightarrow\; f'(0) = 0, \quad f''(0) \neq 0$$

therefore, the multiplicity of the root $x = 0$ is $m = 2$.
(b): Newton's method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} x_n^2}{e^{x_n} x_n (x_n + 2)} = x_n - \frac{x_n}{x_n + 2}.$$

Rate of the convergence is $S = 1 - 1/m = 1 - 1/2 = 1/2$.
(c): Since $m = 2$, we get the following formula for the modified Newton's method;

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} = x_n - \frac{2x_n}{x_n + 2} = \frac{x_n^2}{x_n + 2}.$$

and we calculate $x_1 = 0.004761$ and $x_2 = 0.000015$.

5. Let $f(x) = \cos x + x^2$.

(a) Use the two-point forward difference formula to approximate $f'(0)$ with $h = 0.01$. (5pts)

(b) Use the Richardson's extrapolation for the two-point forward difference formula and derive second order forward difference formula and use it to approximate $f'(0)$ with $h = 0.01$. (10pts)

Solution. (a):

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \quad \Rightarrow \quad f'(0) \approx \frac{f(h)-f(0)}{h} \approx 0.005$$

(b): The two-point forward difference formula is first order formula. So the Richard's extrapolation formula with $n = 1$ gives us

$$f'(x) \approx 2\left(\frac{f(x+h/2)-f(x)}{h/2}\right) - \left(\frac{f(x+h)-f(x)}{h}\right) = \frac{-f(x+h)+4f(x+h/2)-3f(x)}{h}$$

and we approximate $f'(0) \approx -3.124989156333413 \times 10^{-8}$.

6. Solve the system by finding the $PA = LU$ factorization. Show all your steps. (15pts)
$x_1 + 2x_2 - x_3 = -2$
$2x_1 + 4x_2 = -2$
$x_2 - x_3 = -1$
Solution. Step1. The system has $PA = LU$ factorization

$$PA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU$$

Step 2. Solve $LY = PB$ for Y.
$y_1 = -2$
$y_2 = -1$
$0.5y_1 + y_3 = -2 \;\Rightarrow\; y_3 = -1$
Step 3. Solve $UX = Y$ for X.
$2x_1 + 4x_2 = -2 \quad \Rightarrow \quad x_1 = (-2 - 4x_2)/2 = -1$
$x_2 - x_3 = -1 \quad \Rightarrow \quad x_2 = -1 + x_3 = 0$
$-x_3 = -1 \quad \Rightarrow \quad x_3 = 1$

7. Compute the first two iterations of the Jacobi and the Gauss-Seidel methods with starting vector $x^0 = (0, 0, 0)^\top$ for the following linear system. Determine convergence of the methods. (15pts)
$4x_1 - x_2 + x_3 = 1$
$3x_1 + 6x_2 + 2x_3 = 0$.
$3x_1 + 3x_2 + 8x_3 = 4$
Solution. Once we solve i-th equation for x_i, $i = 1, 2, 3$, we get

$$\begin{aligned} x_1 &= \tfrac{1+x_2-x_3}{4} \\ x_2 &= \tfrac{-3x_1-2x_3}{6} \\ x_3 &= \tfrac{4-3x_1-3x_2}{8} \end{aligned} \quad \text{thus, Jacobi iteration is} \quad \begin{aligned} x_1^{k+1} &= \tfrac{1+x_2^k-x_3^k}{4} \\ x_2^{k+1} &= \tfrac{-3x_1^k-2x_3^k}{6} \\ x_3^{k+1} &= \tfrac{4-3x_1^k-3x_2^k}{8} \end{aligned} \quad \text{and Gauss-Seidel is} \quad \begin{aligned} x_1^{k+1} &= \tfrac{1+x_2^k-x_3^k}{4} \\ x_2^{k+1} &= \tfrac{-3x_1^{k+1}-2x_3^k}{6} \\ x_3^{k+1} &= \tfrac{4-3x_1^{k+1}-3x_2^{k+1}}{8} \end{aligned}$$

The first two steps of the Jacobi iteration are

$$x^1 = (0.25, 0, 0.5)^\top, \quad x^2 = (0.125, -0.125, 0.4063)^\top$$

and the first two steps of the Gauss-Seidel method are

$$x^1 = (0.25, -0.125, 0.4531)^\top, \quad x^2 = (-0.0234, -0.0111, 0.4646)^\top$$

Since the coefficient matrix is strictly diagonally dominant the both methods are convergent.