Lecture Summary 2.
Math 241 Sections 20-22, Fall 2008 University of Delaware.

Derivative of a function $y = f(x)$ is

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D_x f = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

The n-th order derivative of a function $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d^n y}{dx^n} = (...(f\underbrace{')')'...)'}_{n\ times} = (f^{(n-1)}(x))'$$

Differentiation Rules. let a and c are constants and f and g are differentiable. Then

$$(c)' = 0, \quad (x^a)' = ax^{a-1}, \quad (cf)' = c(f'), \quad (f \pm g)' = f' \pm g'$$

$$(fg)' = f'g + g'f, \quad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \quad \text{and} \quad (f\circ g)' = f'(g)g'.$$

Derivatives of exponential and logarithm functions. Let $a > 0$ constant.

$$(a^x)' = a^x \ln a, \quad (e^x)' = e^x \quad \text{and} \quad (\log_a x)' = \frac{1}{x\ln a}, \quad (\ln x)' = \frac{1}{x}.$$

$$(e^{g(x)})' = e^{g(x)}g'(x), \qquad \Big(\ln(g(x))\Big)' = \frac{g'(x)}{g(x)}.$$

Logarithmic differentiation:

$$y = f(x) \quad \Rightarrow \quad \ln y = \ln(f(x)) \quad \Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{y'}{y} = \frac{d}{dx}(\ln(f(x)) \quad \Rightarrow \quad y' = f(x)\frac{d}{dx}(\ln(f(x))$$

Derivative of trigonometric functions.

$$(\sin x) = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x = \frac{1}{\cos^2 x},$$

$$(\csc x)' = -\csc x \cot x, \quad (\sec x)' = \sec x \tan x, \quad (\cot x)' = -\csc^2 x = -\frac{1}{\sin^2 x}.$$

Derivative of inverse trigonometric functions.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad (\arctan x)' = \frac{1}{1+x^2}.$$

Hyperbolic Functions.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$

Derivative of hyperbolic functions.

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x, \quad (\tanh x)' = \text{sech}^2 x.$$

The implicit differentiation. Let we have a relation (equation) $F(x,y) = H(x,y)$ that defines a implicit function $y(x)$. To find $y'(x)$;
differentiate both side of the equation with respect to x, regarding $y = y(x)$

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}H(x,y) \quad \Rightarrow \quad F_1(x,y,y') = H_1(x,y,y')$$

then we solve for y'.
The linear approximation of $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x-a) \approx f(x).$$

Differential of $y = f(x)$ is

$$dy = f'(x)dx = L(x) - f(x) \approx \triangle y = f(x+dx) - f(x).$$

Related Rates.

Useful steps:

1. Model your problem.
   - Draw picture or diagram
   - Introduce variables (independent, dependent)
   - Express the given information.
2. Determine a relation between dependent variables.
3. Differentiate the relation with respect to independent variable (The Chain Rule).
4. Substitute the given information into resulting relation and solve for the unknown rate.

Example 1.
If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min. Find the rate at which the radius decreases when the diameter is 10 cm.
Example 2.
The length of a rectangle is increasing at a rate of 8 cm/s and its width increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing ?
Example 3.
A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out ?
Example 4.
A particle is moving along curve $y = \sqrt{x}$. As particle passes through the point $(4,2)$, its x coordination increases at a rate of 3 cm/s. How fast is the distance from particle to the origin changing at this instant ?
Example 5.
The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h How fast is the shadow cast by a 400 ft tall building increasing when the angle of elevation of the sun is $\pi/6$?