Lecture Summary 1.
Math 241 Sections 20-21, Fall 2008 University of Delaware.

Definition. If we can make values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a ($x \neq a$), then we say

"the limit of $f(x)$, as x approaches a, equals L"

and we write

$$\lim_{x \to a} f(x) = L.$$

The right (left) hand limit of $f(x)$, as x approach to a, equals L means; values of $f(x)$ get arbitrarily close to L, when x is taken sufficiently close to a with $x > a$ ($x < a$) and we write

$$\lim_{x \to a^+} f(x) = L \quad (\lim_{x \to a^-} f(x) = L).$$

Infinite limits. If values of $f(x)$ get arbitrarily large (negative large), when x is taken sufficiently close to a ($x \neq a$), then we say the limit of $f(x)$, as x approaches to a, equals infinity (negative infinity) and we write

$$\lim_{x \to a} f(x) = +\infty \quad (\lim_{x \to a} f(x) = -\infty).$$

Limits at infinity. Let f defined on some interval (a, ∞) $((-\infty, a))$. If values of $f(x)$ get arbitrarily close to L by taking x sufficiently large (or negative large), then we say the limit of $f(x)$, as x approach to ∞ $(-\infty)$, is L and we write

$$\lim_{x \to \infty} f(x) = L \quad (\lim_{x \to -\infty} f(x) = L).$$

Limit Laws. Suppose

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

are exist, n is positive integer and c is constant number. Then

$$\mathbf{1}.\ \lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \quad \mathbf{2}.\ \lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

$$\mathbf{3}.\ \lim_{x \to a} (cf)(x) = c \lim_{x \to a} f(x) \quad \mathbf{4}.\ \lim_{x \to a} (f/g)(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x), \text{ if } \lim_{x \to a} g(x) \neq 0$$

$$\mathbf{5}.\ \lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \quad \mathbf{6}.\ \lim_{x \to a} c = c \quad \mathbf{7}.\ \lim_{x \to a} x = a \quad \mathbf{8}.\ \lim_{x \to a} x^n = a^n, \quad \mathbf{9}.\ \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

$$\mathbf{10}.\ \lim_{x \to a} \sqrt[n]{f}(x) = \sqrt[n]{\lim_{x \to a} f(x)} \text{ if n is even we assume limit is nonnegative.}$$

11. If f is a polynomial, rational, root, trigonometric, exponential and logarithm function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Note: All these statements hold true for the right (left) hand limits.
Theorem.

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Theorem (Squeeze). If $f(x) \leq g(x) \leq h(x)$ when x is near a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x), \text{ then } \lim_{x \to a} g(x) = L.$$

Theorem If $a > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^a} = 0.$$

Continuity: A function continuous at a (continuous from left, continuous from right) if

$$\lim_{x \to a} f(x) = f(a) \ \ (\lim_{x \to a^-} f(x) = f(a), \quad \lim_{x \to a^+} f(x) = f(a)).$$

A function continuous on an interval if it is continuous at every number in the interval. When we dealing closed intervals we understand continuous from right (left) at the end point.
Theorem If $f(x)$ and $g(x)$ are continuous at a and c is constant, then following functions are continuous at a

$$(f+g)(x), \ \ (f-g)(x), \ \ (fg)(x), \ \ (cf)(x) \ \text{ and } \ (f/g)(x), \text{ if } g(a) \neq 0.$$

Theorem. If $g(x)$ is continuous at a and $f(x)$ continuous at $g(a)$ then composition function $(f \circ g)(x) = f(g(x))$ is continuous at a.
The Intermediate Value Theorem. If $f(x)$ is continuous on the closed interval $[a, b]$ and M is any value between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = M$.

Discontinuous. If $f(x)$ is not continuous at a then we say $f(x)$ is discontinuous at a.
Removable Discontinuous: If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$ or $f(a)$ is not defined.
Infinite Discontinuous: If $\lim_{x \to a} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.
Jump Discontinuous: If $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ are both exist and difference of the these two limits is finite.
Tangent line for curve $y = f(x)$ at the point $P(a, f(a))$ is $y - f(a) = m(x - a)$, where

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Average rate of change of $f(x)$ with respect to x is

$$Averagerate = \frac{\triangle f}{\triangle x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Instantaneous rate of change of $f(x)$ with respect to x when $x = x_1$ is

$$Instantaneousrate = \lim_{\triangle x \to 0} \frac{\triangle f}{\triangle x} = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}.$$

The derivative of a function f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.