PARTITION OF UNITY FOR THE STOKES PROBLEM ON NONMATCHING GRIDS

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Abstract. We consider the Stokes Problem on a plane polygonal domain \( \Omega \subset \mathbb{R}^2 \). We propose a finite element method for overlapping or nonmatching grids for the Stokes Problem based on the partition of unity method. We prove that the discrete inf-sup condition holds with a constant independent of the overlapping size of the subdomains. The results are valid for multiple subdomains and any spatial dimension.

1. Introduction

In the present literature the study of finite element method applied to overlapping grids is done mainly in the framework of mortar method or Lagrange multiplier (see [1, 12, 7]). Using a partition of unity method which has roots in [2], a new finite element discretization for elliptic boundary value problems was introduced by Huang and Xu in [10]. A significant amount of literature was dedicated to numerical solutions of the Stokes problem (see e.g., [9, 5] and the references of this two books). By our knowledge not to much was done for solving discretization of the Stokes problem when overlapping grids or nonmatching grids are involved. In this paper, following the ideas of Huang and Xu, we shall introduce a conforming finite element method, using a partition of unity type argument for the steady-state Stokes problem.

2. The continuous Stokes problem and overlapping subdomains discretization

Even though the results hold in a more general context and for a general dimension, for clarity, we present the main ideas of the discretization method in case of two subdomains in \( \mathbb{R}^2 \). Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with boundary \( \partial \Omega \) and let \( \Gamma \) be a closed subset of \( \partial \Omega \). By \( H^1_0(\Omega; \Gamma) \) we denote the closure in \( H^1 \)-topology of \( C^\infty(\bar{\Omega}) \) functions that vanish in a neighborhood of \( \Gamma \).

The steady-state Stokes problem in the velocity-pressure formulation is:

Find the vector-valued function \( u \) and the scalar-valued function \( p \) satisfying

\[
\begin{align*}
-\Delta u - \nabla p &= F \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} p &= 0.
\end{align*}
\]

Date: September 10, 2003.

Key words and phrases. non-matching grid, finite element method, partition of unity, Stokes problem.

This work is supported by the NSF Grant No. 0209497 (Multiscale Methods for Partial Differential equations).
Let \((\cdot, \cdot)_\Omega\), or simply \((\cdot, \cdot)\), denote the \(L^2(\Omega)\)-inner product applied to a pair of either scalar or vector functions. Similarly, let \(\|\cdot\|_{0, \Omega}\), or simply \(\|\cdot\|\) denote the \(L^2(\Omega)\)-norm. Define \(V = (H^1_0(\Omega))^2\) and \(P = L^2_0(\Omega)\) the subspace of \(L^2(\Omega)\) consisting of functions with zero mean value on \(\Omega\). The variational formulation of the problem (2.1) is

Find \((u, p) \in (V, P)\) such that

\[
\begin{cases}
a(u, v) + b(v, p) = (F, v) & \text{for all } v \in V, \\
b(u, q) = 0 & \text{for all } q \in P.
\end{cases}
\]

where \(a\) is the Dirichlet form on \(\Omega\) defined by

\[
a(u, v) = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx.
\]

and

\[
b(v, q) = (q, \nabla \cdot v).
\]

We assume that the inf-sup condition

\[
(2.3) \quad c_0 \|p\| \leq \sup_{v \in V} \frac{(p, \nabla \cdot v)}{\|v\|_{1, \Omega}}, \quad \text{for all } p \in P,
\]

holds for a positive constant \(c_0\). Consequently, there is a unique solution \((u, p) \in (V, P)\) of (2.2). Let \(\Omega\) be covered by a family of overlapping subdomains. For a better presentation of the main idea, we consider the case of two overlapping subdomains with polygonal shapes. Let \(\Omega_1, \Omega_2\) be overlapping subdomains of \(\Omega\) satisfying \(\Omega = \Omega_1 \cup \Omega_2\) and \(\Omega_0 = \Omega_1 \cap \Omega_2\). We further assume that \(\Omega_1\) and \(\Omega_2\) are partitioned by quasiuniform finite element triangulations \(T_1\) and \(T_2\) of maximal mesh sizes \(h_1\) and \(h_2\) (which might not match on \(\Omega_0\)). Again, just for the sake of simplicity, we assume that \(\Omega_0\) is a strip-type domain of width \(d = O(h_1)\) and that \(\partial \Omega_0\) is aligned with \(T_1\) and \(T_2\) (see Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{overlapping_grids.png}
\caption{Overlapping grids}
\end{figure}
Next, we let \(\{\phi_1, \phi_2\}\) be a partition of unity subordinate to the covering partition \(\{\Omega_1, \Omega_2\}\) of \(\Omega\), i.e. \(\phi_1, \phi_2\) are continuous functions defined on \(\Omega\) such that \(\phi_1 + \phi_2 = 1, 0 \leq \phi_i \leq 1\), and \(\|\nabla \phi_i\|_{\infty, \Omega} \leq 1/d\). We further assume that \(\phi_1 \equiv 1\) on \(\Omega_1 \setminus \Omega_0\) and \(\phi_1 \equiv 0\) on \(\Omega_2 \setminus \Omega_0\), and \(\phi_2 \equiv 1\) on \(\Omega_2 \setminus \Omega_0\) and \(\phi_2 \equiv 0\) on \(\Omega_1 \setminus \Omega_0\).

To obtain a conforming discretization of the variational problem (2.2) we define first the following spaces

\[ P_{h_i}(\Omega_i) := \{ p \in C^0(\Omega_i) | p|_T \in \mathbb{P}_1, T \in T_i \}, \]

\[ \hat{P}_{h_i}(\Omega_i) := \{ p \in P_{h_i}(\Omega_i) | p = 0 \text{ on } \partial\Omega_i \setminus \partial\Omega \}, \]

\[ V_{h_i}(\Omega_i) := \{ \mathbf{v} \in (H^1_0(\Omega_i; \partial\Omega \cap \partial\Omega_i))^2 | \mathbf{v}|_T \in (\mathbb{P}_1)^2, T \in T_i \}, \]

where, \(\mathbb{P}_1\) denotes the set of polynomials in two variables of degree at most one. Using the above spaces, we are interested in building stable pairs \((V_h, P_h)\), where \(V_h \subset V\) and \(P_h \subset P\), i.e., pairs \((V_h, P_h)\) which satisfy the discrete inf-sup condition

\[ c_0 \| p \| \leq \sup_{\mathbf{v} \in V_h} \left( \frac{(p, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_{1, \Omega}} \right), \text{ for all } p \in P_h. \]

If the above condition is satisfied then the discrete variational problem:

Find \((u_h, p_h) \in V_h \times P_h\) such that

\[ \begin{cases} a(u_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{F}, \mathbf{v}) & \text{for all } \mathbf{v} \in V_h, \\ b(u_h, q) = 0 & \text{for all } q \in P_h, \end{cases} \]

has unique solution and the error satisfies,

\[ |u - u_h|_{1, \Omega} + \|p - p_h\|_{0, \Omega} \leq C \left( \inf_{\mathbf{v}_h \in V_h} |u - \mathbf{v}_h|_{1, \Omega} + \inf_{q_h \in P_h} \|p - q_h\|_{0, \Omega} \right), \]

with \(C\) depending on \(c_0\), but independent of \(h\) (or the spaces \(V_h\) and \(P_h\)). In the next two sections we build stable pairs \((V_h, P_h)\) which have also good approximation properties.

3. First mini-type stable pair

We introduce a space \(B\) of bubble functions associated with the “union” partition \(T := T_1 \cup T_2\) as follows. For a triangle \(T\) we define the bubble function \(B_T\) supported on \(T\) as the product of the nodal functions associated with the vertices of \(T\).

If \(K = T_1 \cap T_2 \in T_1 \cup T_2\) we define

\[ B_K := B_{T_1} \cdot B_{T_2}. \]

If \(K = T_i\) for some \(T_i \in T_i\), \((i = 1, 2)\), then we just take \(B_K := B_{T_i}\) (see Fig. 2). A composite, conforming finite element space for velocity can be defined by

\[ V_h \equiv V_h(\Omega) := \phi_1 V_{h_1} + \phi_2 V_{h_2} + B^2. \]

The discrete pressure space we associate with \(V_h\) is

\[ P_h := (\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)) \cap P. \]

Let \(h := h_1 \geq h_2 = rh_1\), for some positive constant \(r\). Before we state the main result of this section we introduce the following assumption:
• (A1) There exists a positive constant $c$ such that $|K| \cong ch^2$ for any $K \in T$,
where $|K|$ denotes the Lebesgue measure of $K \in T$.

**Theorem 3.1.** If (A1) is satisfied, then the pair $(V_h, P_h)$ defined above is a stable pair.

**Proof.** We will construct two operators $\Pi_1 : V \to V_h$, $\Pi_2 : V \to V_h$ with the following properties:

\[
|v - \Pi_1 v|_{1, \Omega} \lesssim |v|_{1, \Omega}, \quad \text{for all } v \in V,
\]

(3.1)

\[
|\Pi_2(I - \Pi_1)v|_{1, \Omega} \lesssim |v|_{1, \Omega}, \quad \text{for all } v \in V,
\]

(3.2)

\[
b(v - \Pi_2 v, q) = 0, \quad \text{for all } q \in P_h, v \in V.
\]

(3.3)

Having constructed $\Pi_1$ and $\Pi_2$, the operator $\Pi_h = \Pi_1 + \Pi_2(I - \Pi_1)$ satisfies the the hypothesis of Proposition 2.8 in [5], for example, and the inf-sup condition follows according with this Proposition.

For $i = 1, 2$, let $V_i := (H^1_0(\Omega_i; \partial \Omega_i \cap \partial \Omega_i))^2$ and define $\Pi_i^1 : V_i \to V_{h_i}$ to be good regularization operators. For example, we can take $\Pi_i^1$ to be Clement-type operators. Thus,

\[
\|v - \Pi_i^1 v\|_{0, \Omega_i} \lesssim h_i|v|_{1, \Omega_i}, \quad \text{for all } v \in V_i,
\]

(3.4)

and

\[
|v - \Pi_1^1 v|_{1, \Omega_i} \lesssim |v|_{1, \Omega_i}, \quad \text{for all } v \in V_i.
\]

(3.5)

We define $\Pi_1$ as follows:

\[
\Pi_1 v := \phi_1 \Pi_1^1(v|_{\Omega_1}) + \phi_2 \Pi_1^2(v|_{\Omega_2})
\]
Note that \( v_{|\Omega_i} \in V_i \) and \( \Pi_1v \in V_h \). Thus \( \Pi_1 \) is well defined. In order to simplify the notation we denote \( \Pi_1^1(v_{|\Omega_i}) \) simply by \( \Pi_1^1v \). Next, we verify that the operator \( \Pi_1 \) satisfies (3.1). We will prove first that the notation we denote \( \Pi_1 \)

\[
\|v - \Pi_1 v\|_{0,\Omega} \lesssim h|v|_{1,\Omega}, \quad \text{for all } v \in V.
\]

Indeed,

\[
\|v - \Pi_1 v\|_{0,\Omega} = \left\| \sum_{i=1}^{2} \phi_i(v - \Pi_1^i v) \right\|_{0,\Omega} \leq \left\| \sum_{i=1}^{2} \phi_i(v - \Pi_1^i v) \right\|_{0,\Omega_i} \leq \sum_{i=1}^{2} \|v - \Pi_1^i v\|_{0,\Omega_i} \leq \sum_{i=1}^{2} h_i |v|_{1,\Omega_i} \lesssim h|v|_{1,\Omega}.
\]

The justification of (3.1) is also straightforward.

\[
|v - \Pi_1 v|_{1,\Omega} \leq \sum_{i=1}^{2} |\phi_i(v - \Pi_1^i v)|_{1,\Omega_i} \leq \sum_{i=1}^{2} |\nabla \phi_i(v - \Pi_1^i v)|_{0,\Omega_i} + \sum_{i=1}^{2} |v - \Pi_1^i v|_{1,\Omega_i} \lesssim d^{-1} \sum_{i=1}^{2} |v - \Pi_1^i v|_{0,\Omega_i} + \sum_{i=1}^{2} |v|_{1,\Omega_i} \lesssim |v|_{1,\Omega}.
\]

Next, we define \( \Pi_2 \). For \( v \in V \) and \( K \in T \) define

\[
\Pi_2 v_{|K} := \alpha B_K,
\]

where \( \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \) is determined such that \( \int_K (v - \Pi_2 v) \, dx = 0 \) i.e.,

\[
\alpha = \frac{\int_K v \, dx}{\int_K B_K \, dx}.
\]

For \( v \in V \) and \( q \in P_h \) we have

\[
b(v - \Pi_2 v, q) = -(v - \Pi_2 v, \nabla q) = -\sum_{K \in T} \nabla q \int_K (v - \Pi_2 v) \, dx = 0.
\]

Thus (3.3) holds and to end the proof we have to verify that (3.2) holds. Let us note that

\[
|\Pi_2 v|_{1,K} \lesssim h^{-1}|v|_{0,K} \quad \text{for all } v \in V.
\]

The proof of (3.7) is a consequence of the following two estimates.

\[
|\Pi_2 v|_{1,K}^2 = (\alpha_1^2 + \alpha_2^2) \int_K |\nabla B_K|^2 \lesssim |\alpha|^2|K|h^{-2},
\]

and

\[
|\alpha|^2 = \frac{\int_K v \, dx|^2}{\int_K B_K \, dx^2} \lesssim \frac{|K||v|_{0,K}^2}{h^4}.
\]

Thus, from (3.7) and (3.6) we obtain

\[
|\Pi_2(I - \Pi_1)v|_{1,\Omega}^2 = \sum_{K \in T} |\Pi_2(I - \Pi_1)v|_{1,K}^2 \lesssim \sum_{K \in T} h^{-2}|(I - \Pi_1)v|_{0,K}^2 \lesssim |v|_{1,\Omega}^2,
\]

which proves that (3.2) holds and concludes the proof of the theorem. \( \square \)
Remark 3.1. In the special case when any $K \in \mathcal{T}$ is a triangle in either $T_1$ or $T_2$ (any $T_1 \in \mathcal{T}_1$, $T_1 \subset \Omega_0$ is an union of triangles of $\mathcal{T}_2$ and any $K \in \mathcal{T}$ which is not subset of $\Omega_0$ belongs to either $T_1$ or $T_2$), we have that $(A1)$ is satisfied. Moreover we have

- $(A1)'$ There exists a positive constant $c$ such that
  $$|T_i| \approx c h_i^2 \text{ for any } K = T_i \subset \mathcal{T}. $$

Following the proof of the above theorem in this particular case, we deduce that the constants which are involved in (3.1) and (3.2) are also independent of the ratio $r = h_2/h_1$. Consequently, the inf-sup condition holds with a constant independent of $h_2, h_1$ and $r$.

Remark 3.2. According with [10] the space $V_h$ has the following approximation property:

$$\inf_{v_h \in V_h} \|v - v_h\|_{1, \Omega} \lesssim h_1\|v\|_{2, \Omega_1} + h_2\|v\|_{2, \Omega_2}, \quad \text{for all } v \in (H^2(\Omega) \cap H^1_0(\Omega))^2. $$

If $\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)$ is a linear space which contains the constant function then, by standard finite element tools, we have that the space $P_h$ has the following approximation property

$$\inf_{p_h \in P_h} \|p - p_h\|_{0, \Omega} \lesssim h_1\|p\|_{1, \Omega} + h_2\|p\|_{1, \Omega}, \quad \text{for all } p \in H^1(\Omega) \cap L^2_0(\Omega).$$

Therefore the pair $(V_h, P_h)$ has good approximation properties and is a stable pair. On the other hand, if $\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)$ is a linear space which does not have good approximation properties we can consider for the discrete pressure space $P_h$ a partition of unity type space and modify accordingly the velocity space. This is the subject of the next section.

4. SECOND MINI-TYPE STABLE PAIR

A discrete pressure space $P_h$ with good approximation properties (see [10]) is the space

$$P_h := (\phi_1 P_{h_1}(\Omega_1) + \phi_2 P_{h_2}(\Omega_2)) \cap P. $$

Since the pressure space is enriched (on the overlapping region), in order to have satisfied the inf-sup condition, we have to enrich the velocity space also. As in the previous section we define a bubble space $B$. For each $K \in \mathcal{T}$, $K \subset \Omega_0$ we let $B^j_K, j = 1, 2$ to be two bubble functions supported on $K$ which have certain properties and are to be specified later. For each one of the remaining regions $K \in \mathcal{T}$ we consider only one bubble function defined as in the first case. We let $B$ to be the span of all these bubble functions and define the discrete space $V_h$ as before:

$$V_h := \phi_1 V_{h_1} + \phi_2 V_{h_2} + B^2. $$

Theorem 4.1. If $(A1)$ is satisfied, then the new pair $(V_h, P_h)$ defined above is a stable pair.

Proof. We follow the construction procedure revealed in Theorem 3.1. The $\Pi_1$ operator is the one defined in the proof of Theorem 3.1. Next, we define $\Pi_2$ such that (3.2) and (3.3) are satisfied. Let $\phi_1 := \phi$ and $\phi_2 := 1 - \phi$. To simplify the computation we will assume that $\phi$ is a linear function in only one variable, say $x$. Thus, for any $q \in P_h$ and any $K \in \mathcal{T}$, $K \subset \Omega_0$ we have that

$$\nabla q_{i,k} \in \text{span} \left \{ \left( \frac{x}{0} \right), \left( \frac{y}{x} \right), \left( \frac{1}{0} \right), \left( \frac{0}{1} \right) \right \} = \text{span} \left \{ \left( \frac{x - x_0}{0} \right), \left( \frac{y - y_0}{x - x_0} \right), \left( \frac{1}{0} \right), \left( \frac{0}{1} \right) \right \} ,$$
where \((x_0, y_0)\) are the coordinates of any point inside \(K\). We define \(\Pi_2\) as follows
\[
\Pi_2 v|_K := \left( \frac{\alpha_1}{\alpha_2} \right) B_K^1 + \left( \frac{\beta_1}{\beta_2} \right) B_K^2,
\]
if \(K \in T, K \subset \Omega_0\) and \(\Pi_2 v|_K := \left( \frac{\alpha_1}{\alpha_2} \right) B_K\) if \(K \in T\) and \(K \subset \Omega_i \setminus \Omega_0, i = 1, 2\). The constants \(\alpha = \left( \frac{\alpha_1}{\alpha_2} \right)\) and \(\beta = \left( \frac{\beta_1}{\beta_2} \right)\) are determined such that \(\int_K (v - \Pi_2 v) \cdot \nabla q \, dx = 0, q \in P_h\). Here, in the second case, \(B_K\) is the bubble function defined in the previous section. Obviously, (3.3) holds. The justification of (3.2) is similar and we only need to prove that (3.7) holds. For \(K \in T\) and \(K \subset \Omega_i \setminus \Omega_0, i = 1, 2\) the proof was done in the previous section. We will focus now on the case \(K \in T, K \subset \Omega_0\). From the definition of \(\Pi_2\) and the condition \(\int_K (v - \Pi_2 v) \cdot \nabla q \, dx = 0\) we deduce that
\[
\begin{align*}
\alpha_1(B_K^1, x - x_0) + \beta_1(B_K^2, x - x_0) &= (v_1, x - x_0) \\
\alpha_1(B_K^1, y - y_0) + \beta_1(B_K^2, y - y_0) + \alpha_2(B_K^1, x - x_0) + \beta_2(B_K^2, x - x_0) &= (v_1, y - y_0) + (v_2, x - x_0) \\
\alpha_1(B_K^1, 1) + \beta_1(B_K^2, 1) &= (v_1, 1) \\
\alpha_2(B_K^1, 1) + \beta_2(B_K^2, 1) &= (v_2, 1),
\end{align*}
\]
where \(v_1, v_2\) are the components of the velocity vector \(v\) restricted to \(K\). The system has unique solution if and only if
\[
det_K := det \left( \begin{bmatrix} B_K^1, x - x_0 \\ B_K^2, x - x_0 \end{bmatrix} \right) \neq 0.
\]
Let us assume that \(B_K^1\) and \(B_K^2\) are chosen such that (4.2) is satisfied. Then one can solve for \(\alpha\) and \(\beta\). For example we have
\[
\alpha_1 = \frac{1}{det_K} \left( (B_K^2, 1)(v_1, x - x_0) - (B_K^2, x - x_0)(v_1, 1) \right).
\]
If we further assume that
\[
|\nabla B_K^i| \lesssim h^{-1} \text{ on } K.
\]
Then, using (A1), we get
\[
|\Pi_2 v|_1^2, K \leq (\alpha_1^2 + \alpha_2^2) \int_K |\nabla B_K^1|^2 + (\beta_1^2 + \beta_2^2) \int_K |\nabla B_K^2|^2 \lesssim (|\alpha|^2 + |\beta|^2).
\]
On the other hand from (4.3) and the fact that \(|x - x_0| \leq h\) we have that
\[
|\alpha_1| \leq \frac{1}{|det_K|} h^2 |v_1|_{0,K} h^2.
\]
If \(B_K^1\) and \(B_K^2\) are chosen such that we have
\[
|det_K| \geq h^5, \quad \text{for all } K \in T, K \subset \Omega_0,
\]
then
\[
|\alpha_1| \lesssim h^{-1} |v|_{0,K},
\]
and similar estimates hold for \(\alpha_2, \beta_1\) and \(\beta_2\). Hence, via (4.5), we have that (3.7) holds and consequently (3.2) is satisfied. All we have left to do is to justify that we can choose \(B_K^1\) and \(B_K^2\) such that (4.4) and (4.6) are satisfied.
One way of choosing $B_{K}^{1}$ and $B_{K}^{2}$ is as follows. First we define $R = R_{K}$ to be a rectangle inscribed in $K$ of area of order $h^{2}$ and with the center of symmetry at $(x_{0}, y_{0})$. Without loss of generality we may assume that $R$ has the sides parallel with the system of coordinates where $\Omega$ lies. We denote by $R_{1} = \{(x, y) \in R_{K} | x < x_{0}\}$ and $R_{2} = \{(x, y) \in R_{K} | x > x_{0}\}$. Next, we define $B_{K}$ to be the bubble function associated with the rectangle $R$. Then,

$$B_{K}^{1} := \frac{1}{h}(x_{0} - x)B_{K} \cdot \chi_{R_{1}} \text{ and } B_{K}^{2} := \frac{1}{h}(x_{0} - x)B_{K} \cdot \chi_{R_{2}},$$

where $\chi_{R}$ stands for the characteristic function of the set $R$. One can easily check that (4.4) and (4.6) are satisfied for this choice of $B_{K}^{1}$ and $B_{K}^{2}$.

5. Nonoverlapping Nonmatching grids

Let $\Omega$ be split into two nonoverlapping subdomains $\Omega_{1}$ and $\Omega_{2}$ and let $\Gamma$ be the interface between $\Omega_{1}$ and $\Omega_{2}$. As in the overlapping case we assume that $\Omega_{1}$ and $\Omega_{2}$ are partitioned by quasiuniform finite element triangulations $T_{1}$ and $T_{2}$ of maximal mesh sizes $h_{1}$ and $h_{2}$ ($h_{1} \geq h_{2}$). The grid on the interface $\Gamma$ might not match the two partitions. We can extend the mesh of $\Omega_{2}$ inside $\Omega_{1}$ so that the overlapping meshed region is a strip $\Omega_{0}$ of size $d = O(h_{1})$ which matches the mesh of $\Omega_{1}$. In this way we have reduced the setting to the case of overlapping subdomains. Nevertheless, if the extension is not careful done, the condition (A1) might not be satisfied (or poorly satisfied). One way to construct a good extension is by slightly modify the mesh on one subdomain. Next, we describe a way of constructing such a grid extension. Let us assume that the mesh of $\Omega_{2}$ is extended inside $\Omega_{1}$ for a “strip” with $d = O(h_{1})$ (see Fig. 3). Let $\Gamma_{1}$ be the border of our extension which lies inside $\Omega_{1}$. We assume that $\Gamma_{1}$ matches the mesh on $\Omega_{1}$ and ignore the mesh $T_{1}$ between $\Gamma_{1}$ and $\Gamma$. We present in Fig. 3 a case with $h_{1}/h_{2} = r = 11/4$.

If $A \in \Gamma_{1}$ is an arbitrary nodal point in $T_{1}$, we first connect $A$ with the closest nodal point in $T_{2}$ situated on $\Gamma$ (point $B$ in our picture). After all “horizontal connections” are made we subdivide all the quadrilaterals new formed by connecting two diagonal points. This produces a new mesh on $\Omega_{0}$ which together with the unchanged mesh on $\Omega_{1}$ creates a new mesh on $\Omega_{1}$ denoted $\tilde{T}_{1}$. Next we refine the triangulation on $\Omega_{0}$ to a quasi-uniform triangulation of size $h_{2}$ which matches the mesh on $\Omega_{2}$ (not showed in our picture).

Let $\Omega_{2}$ be the domain made up by $\Omega_{2}$ and $\Omega_{0}$ and let $\tilde{T}_{2}$ be the mesh which extends $T_{2}$ with the fine mesh on $\Omega_{0}$ of size $h_{2}$. We note that the meshes $\tilde{T}_{1}$ and $\tilde{T}_{2}$ corresponding to the subdomain $\Omega_{1}$ and $\tilde{\Omega}_{2}$ respectively, satisfy the special case presented in Remark 3.1. Therefore, the mini-type stable pair presented in Section 3 can be involved for solving the discrete Stokes problem if nonmatching grids are provided.

6. Conclusions

- The method can be extended with no difficulties to the more subdomains case or the multidimensional case.
- If the discrete approximation spaces are spaces of continuous piecewise linear functions then the partition of unity functions can be chosen to be piecewise linear functions also.
• The condition (A1) is too restrictive. In practice, we can slightly change the mesh by moving points of the mesh towards other close points or edges.
• We conjecture that other classical stable pairs for subdomains, for example $\mathbb{P}_2-\mathbb{P}_1$ “subspaces”, could be glued by partition of unity method in order to construct stable pairs with good approximation properties.

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