Notes on the Schwartz Alternating Method for Partition of Unity FEM

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Abstract. We consider discretization on overlapping non-matching grids for elliptic equations by using the Schwartz alternating (SA) method. We investigate also the dependence between the angle of partition of unity (PU) subspaces and the condition number of the stiffness matrix for a model problem. The aim of the paper is to find strategies to choose optimal or quasi-optimal partition of unity set of functions for PU discretizations for elliptic problems on overlapping non-matching grids.

Keywords. Partition of Unity, Finite Element Method, Overlapping Grids, Schwartz Alternating Method.

AMS (MOS) subject classification: 65N30.

1 Introduction

The partition of unity finite element method (PUFEM) was first proposed by Melenk and Babuska \([8]\). As one of the meshless methods, the PUFEM has the ability to include a prior knowledge and to construct finite element spaces of any desired regularity. Since then, it has been applied to treat various problems including the elastically supported beam, bimaterial interface cracks, linear diffusion and convection problems, etc. e.g., see \([2, 10, 9]\). Extension and analysis of the PUFEM can be found, but not restrict to, in \([5, 7]\).

The next sections provides the basic concepts needed for this paper. In Section 3, we study the dependence between the \(L^2\) strengthened Cauchy-Schwart (SCS) constant of two PU subspaces and the condition number of the global stiffness matrix for a model elliptic problem. In section 4 we investigate the optimality of a Schwartz alternating method for two subspaces in terms of the choice of the PU functions. This study is based on the strengthened Cauchy-Schwart (SCS) constant in the energy norm of two PU subspaces. Numerical results to support the PU approach for thin overlap are presented in the end.

Let \(\{\phi_i\}\) be a Lipschitz partition of unity subordinate to the cover \(\{\Omega_i\}\) satisfying

\[
\text{supp}\phi_i \subset \text{closure}(\Omega_i) \quad \forall i,
\]

\[
\sum_i \phi_i \equiv 1 \quad \text{on} \quad \Omega,
\]

\[
\|\phi_i\|_{L^\infty(\mathbb{R}^n)} \leq C_\infty,
\]

\[
\|\nabla \phi_i\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_G}{d_{\text{iam}} \Omega_i},
\]

where \(C_\infty, C_G\) are two constants. Then \(\{\phi_i\}\) is called a \((M, C_\infty, C_G)\) partition of unity subordinate to the cover \(\{\Omega_i\}\). The partition of unity \(\{\phi_i\}\) is said to be of degree \(m \in N_0\) if \(\phi_i \in C^m(\mathbb{R}^n)\). The covering sets \(\{\Omega_i\}\) are called patches.

Definition 2.2 Let \(\{\Omega_i\}\) be an open cover of \(\Omega \subset \mathbb{R}^n\) and let \(\{\phi_i\}\) be a \((M, C_\infty, C_G)\) partition of unity subordinate to \(\{\Omega_i\}\). Let \(V_i \subset H^1(\Omega_i \cap \Omega)\) be given. Then the space

\[
V := \sum_i \phi_i V_i = \{ \sum_i \phi_i v_i | v_i \in V_i \} \subset H^1(\Omega)
\]

is called the PUFEM space. The PUFEM space \(V\) is said to be of degree \(m \in N_0\) if \(V \subset C^m(\Omega)\). The space \(V_i\) are referred to as the local approximation spaces.

In \([7]\), The authors consider the thin overlapping regions. Let each \(\Omega_i\) is partitioned by quasi-uniform triangulation \(\tau^{h_i}\) of maximal mesh size \(h_i\). With each triangulation \(\tau^{h_i}\), associate a finite element spaces \(V_i \subset H^r(\Omega_i)\). Let \(u \in H^r(\Omega)\), and let \(m_i \geq 1\) denote the additional degree of smoothness of \(u\) on \(\Omega_i\). Assume optimal approximation property on subdomains: For any \(u \in H^{m_i+r}(\Omega_i)\), there exist a \(v_h \in V_i\) such that

\[
\sum_{k=0}^r h_i^k |u - v_h|_{k, \Omega_i} \leq c h_i^{m_i+r} \|u\|_{m_i+r, \Omega_i}.
\]

Also assume that

\[
|\nabla^k \phi_i| \leq c d_i^{-k}, 1 \leq k \leq r
\]

where \(d_i\) is the minimal overlapping size of \(\Omega_i\) with its neighboring subdomains.

Theorem 2.1 (Huang-Xu, 2002) If the overlapping size \(d_i \geq c h_i\), then for \(0 \leq k \leq r\),

\[
\inf_{v_h \in V_i} \|u - v_h\|_{k, \Omega_i} \leq C \sum_{i=1}^p h_i^{m_i+r-k} \|u\|_{m_i+r, \Omega_i},
\]

for any \(u \in H^r(\Omega) \cap \bigcap_{i=1}^p H^{m_i+r}(\Omega_i)\).
For $u \in H^2(\Omega)$ and $H^1$ conforming finite element space, we have
\[
\inf_{v_h \in V} \|u - v_h\|_0,\Omega \leq C \sum_{i=1}^{p} h_i^2 \|u\|_{2,\Omega_i},
\]
\[
\inf_{v_h \in V} \|u - v_h\|_{1,\Omega} \leq C \sum_{i=1}^{p} h_i \|u\|_{2,\Omega_i},
\]
where we set $k = 0$ and $k = 1$ in the above theorem and $m_i = 1$, $r = 1$.

The Schwartz’s alternating method can be found in most domain decomposition books or finite element books (e.g., [6]). Consider the variational problem
\[
a(u, v) = \langle f, v \rangle \quad \forall v \in H. \tag{2.1}
\]
Here $a(\cdot, \cdot)$ is the inner product of the Hilbert space $H$ and $\| \cdot \|$ is the corresponding norm. Let $H$ be the direct sum of two subspaces
\[
H = V \oplus W, \tag{2.2}
\]
and assume that solving (2.1) on either $V$ or $W$, is a much easier task than solving the whole problem. Let $u_0 \in H$. When $u_{2i}$ is already determined, find $v_{2i} \in V$ such that
\[
a(u_{2i} + v_{2i}, v_{2i}) = \langle f, v \rangle \quad \forall v \in V. \tag{2.3}
\]
Set $u_{2i+1} = u_{2i} + v_{2i}$. When $u_{2i+1}$ is already determined, find $w_{2i+1} \in W$ such that
\[
a(u_{2i+1} + w_{2i+1}, w_{2i+1}) = \langle f, w \rangle \quad \forall w \in W. \tag{2.4}
\]
Then set $w_{2i+2} = w_{2i+1} + w_{2i+1}$.

The so-called strengthened Cauchy inequality is crucial in the analysis.

**Theorem 2.2 Convergence Theorem.** Assume that there is a constant $\gamma \leq 1$ such that for the inner product in $H$
\[
|a(v, w)| \leq \gamma \|v\| \|w\| \quad \text{for } v \in V, w \in W. \tag{2.5}
\]
Then, for the Schwartz alternating iteration, we have that the error reduction is given by
\[
\|u_{k+1} - u\| \leq \gamma \|u_k - u\| \quad \text{for } k \geq 1. \tag{2.6}
\]
A proof of the theorem can be found in [6]. The constant $\gamma$ is often called the strengthened Cauchy-Schwartz (SCS) constant (in the energy inner product).

### 3 The SCS Constant

In this section, we consider the SCS constant (in $L^2$-inner product) for two PU finite element spaces and investigate its relation with the condition number of the global stiffness matrix for the model elliptic problem. For simplicity, we consider two overlapping subdomains $\Omega_1$ and $\Omega_2$. To make our presentation more precise, we define the following PUFEM subspace.

**Definition 3.1** Let $\{\Omega_i\}, i = 1, 2$ be an open cover of $\Omega \subset \mathbb{R}^n$ and let $\{\phi_i\}, i = 1, 2$ be a $(M, C_\infty, C_G)$ partition of unity subordinate to $\{\Omega_i\}, i = 1, 2$. Let $V_i \subset H(\Omega_i \cap \Omega), i = 1, 2$ be given. Then the space
\[
V_i := \{\phi_i v_j | v_j \in V_i \subset H(\Omega), \ i = 1, 2
\]
is called the PUFEM subspace. Note $H(\Omega)$ is a Hilbert space of functions defined on $\Omega$.

**Remark 3.1** From the computational point of view, it is desirable to have the linear independence of the collection of all PU base functions associated with the subdomains. For thin overlapping case when the boundary of one mesh lies inside the other mesh (it is not aligned with the other mesh), it is easy to justify that, that $\phi_1 V_1 \cap \phi_2 V_2 = \{0\}$. This gives the linear independence of the entire collection of all PU base functions. When the meshes are more regular on the overlapping region, the linear independence property for two sub-domains was studied in [3].

Let $\Omega_1$, $\Omega_2$ be an overlapping covering of a two dimensional polygonal domain (see Figure 1). We assume that $\Omega_1$ and $\Omega_2$ are partitioned by quasi-uniform finite element triangulations of maximal mesh size $h_1$ and $h_2$, and $\Omega_0 = \Omega_1 \cap \Omega_2$ is a strip-type domain of width $d = O(h_1)$. Let $\{\phi_1, \phi_2\}$ be a PU subordinated to the

![Figure 1: Overlapping Meshes in $\Omega$.](image-url)
If $V_k$’s are spaces of functions with no zero boundary conditions, it is enough to take $\phi_1, \phi_2$ to be piecewise quadratic. To estimate the condition number of the PUFEM space described in the above theorem we need to introduce the SCS angle $\gamma$ in the $L^2$ inner product of $\phi_1 V_1$ and $\phi_2 V_2$. Let $\gamma \in (0, 1)$ be the SCS constant (the cosine of the angle between the subspaces $\phi_1 V_1$ and $\phi_2 V_2$) in the $L^2$ inner product, i.e.,
\[
\gamma := \sup_{u_i \in \phi_i V_i} \frac{\|u_1 \cdot u_2\|}{\|u_1\| \cdot \|u_2\|} \leq 1.
\]

**Theorem 3.3** Suppose that $V_k, k = 1, 2$ are finite element spaces of continuous piecewise linear functions which are zero on the boundary of $\Omega \cap \Omega_k$ and $\phi_1, \phi_2$ are piecewise linear partition of unity functions subordinated to the covering $\Omega_1 \cup \Omega_2 = \Omega$, and satisfy (3.7). Let $V = \phi_1 V_1 + \phi_2 V_2$ and consider the problem: Find $u \in V$ such that
\[
a(u, v) = (f, v) \text{ for all } v \in V, \quad (3.8)
\]
where
\[
a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \; dx,
\]
\[
(f, v) := \int_{\Omega} f \cdot v \; dx.
\]

Let $A$ be the stiffness matrix associated with form $a(\cdot , \cdot)$ and the PU nodal basis functions of Theorem 3.2. Assume that $h_1, h_2$ are the mesh sizes for $T_1$ on $\Omega_1$ and $T_2$ on $\Omega_2$, respectively, and that $h_1 \geq h_2$. Then,
\[
c_1 \frac{1}{h_1^2} \leq \mathcal{K}(A) \leq c_2 \frac{1}{1 - \gamma} \frac{1}{h_2^2},
\]
where $\mathcal{K}(A)$ is the condition number of the matrix $A$, and $c_1, c_2$ are two constants independent of $\gamma, h_1, h_2$.

**Proof** First we notice that, by using the CS and the SCS inequalities, we have
\[
(1 - \gamma)(\|u_1\|^2 + \|u_2\|^2) \leq \|u_1 + u_2\|^2 \leq 2 (\|u_1\|^2 + \|u_2\|^2),
\]
for all $u_k \in \phi_k V_k, k = 1, 2$, where $\| \cdot \|$ denotes the standard $L^2$ inner product. To estimate the condition number we follow the same ideas presented in [4], Chapter 2. Let $\{\phi_1 \psi_i\}_{i=1, \ldots, n_1}$ and $\{\phi_2 \eta_j\}_{j=1, \ldots, n_2}$ be bases for $\phi_1 V_1$ and $\phi_2 V_2$, respectively. Then, according to Theorem 3.2, we have that $\{\varphi_k\}_{k=1, \ldots, n+n_2} := \{\phi_1 \psi_i\}_{i=1, \ldots, n_1} \cup \{\phi_2 \eta_j\}_{j=1, \ldots, n_2}$ is a basis for $V = \phi_1 V_1 + \phi_2 V_2$ and the stiffness matrix $A$ are $A_{ij} = a(\psi_i, \psi_j), i, j = 1, \ldots, n$, where $n = n_1 + n_2$, $\phi_i = \phi_1 \psi_i$ for $i = 1, \ldots, n_1$ and $\varphi_{n_1+j} = \phi_2 \eta_j$ for $j = 1, \ldots, n_2$. Thus, if $u = \sum_{k=1}^{n+n_2} c_k \varphi_k \in V$, and $c := (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, then $\mathcal{K}(A) = \lambda_{max}(A) / \lambda_{min}(A)$, where
\[
\lambda_{max}(A) = \sup_{c \in \mathbb{R}^n} \frac{(Ac, c)}{(c, c)} = \sup_{u = \sum_{k=1}^{n+n_2} c_k \varphi_k \in V} \frac{a(u, u)}{(c, c)}.
\]
and
\[
\lambda_{min}(A) = \inf_{c \in \mathbb{R}^n} \frac{(Ac, c)}{(c, c)} = \inf_{u = \sum_{k=1}^{n+n_2} c_k \varphi_k \in V} \frac{a(u, u)}{(c, c)}.
\]

Here $(\cdot, \cdot)$ is the standard Euclidian inner product and the inf and the sup are taken over non-zero vectors. In what follows, $C, C_1, C_2$ are constants independent of $\gamma, h_1$ and $h_2$ and might be different at different occurrences. Using that $\phi_1$ can be taken identical 1 on $\Omega_1 \cap \Omega_2$, and that on the overlapping region above any triangle $T_1 \subset T_2$ the function $\phi_1$ is a linear combination of the nodal functions associated with the vertices of $T_1$ (with coefficients “independent” of $T_1$) we have that there are constants $C, C_1, C_2$ such that
\[
C_1 \|u_1\|^2 \leq h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 \leq C_2 \|u_1\|^2
\]
for all $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and
\[
a(\phi_2 \eta_j, \phi_2 \eta_j) \leq C, \quad j = 1, \ldots, n_2.
\]

Similar arguments for the subspace $V_2$ lead to
\[
C_1 \|u_2\|^2 \leq h_2^2 \sum_{j=1}^{n_2} \beta_j^2 \leq C_2 \|u_2\|^2
\]
for all $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$ and
\[
a(\phi_2 \eta_j, \phi_2 \eta_j) \leq C, \quad j = 1, \ldots, n_2.
\]

Let $u = u_1 + u_2 \in V$, where $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$. Then, using (3.14), (3.16) and the finite interaction of the nodal functions, we get
\[
\begin{align*}
a(u, u) &\leq 2(a(u_1, u_1) + a(u_2, u_2)) \\
&\leq C \sum_{i=1}^{n_1} \alpha_i^2 a(\psi_i \psi_i) + \sum_{j=1}^{n_2} \beta_j^2 a(\eta_j \eta_j) \\
&\leq C \sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2.
\end{align*}
\]

From the above estimate and (3.12) we get that
\[
\lambda_{max}(A) \leq C.
\]

On the other hand, from the Poincare’s inequality, (3.13), and (3.15) we have
\[
\begin{align*}
a(u, u) &\geq C \|u\|^2 \geq C (1 - \gamma) \|u_1\|^2 + \|u_2\|^2 \\
&\geq C (1 - \gamma) (h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 + h_2^2 \sum_{j=1}^{n_2} \beta_j^2) \\
&\geq C (1 - \gamma) h_2^2 \sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2.
\end{align*}
\]

From (3.11) we get that
\[
\lambda_{min}(A) \geq C (1 - \gamma) h_2^2.
\]
Hence, we have proved the left side of (3.9). Next, we prove the right side of (3.9). Let $u = \phi_1 \psi_1$. The coefficient vector associated with $u$ is $e_i = (1, 0, ..., 0)^T \in \mathbb{R}^n$, and using the locality of $\psi_1$ and the properties of $\phi_1$ we get

$$a(u, u)_{\{e_1, e_1\}} = C.$$  

From (3.12) we get that

$$\lambda_{\text{max}}(A) \geq C. \quad (3.17)$$

To find an upper bound for $\lambda_{\text{min}}(A)$ we fix a non-zero function $u \in H^1_0$. According with Theorem 2.1 there exists $u_h \in V$ such that

$$||u - u_h|| \leq C(h_1||u||_{H^1(\Omega_1)} + h_2||u||_{H^1(\Omega_2)}), \quad (3.18)$$

and

$$a(u_h, u_h) \leq C a(u, u). \quad (3.19)$$

From (3.18) we get

$$||u_h|| \geq ||u|| - ||u - u_h|| \geq 1/2||u|| = C_1, \quad (3.20)$$

if $h = h_1$ is small enough. Assume that $u_h = u_1 + u_2$, where $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$. Then we have

$$C_1^2 \leq ||u_h||^2 \leq 2(||u_1||^2 + ||u_2||^2) \leq C (h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 + h_2^2 \sum_{j=1}^{n_2} \beta_j^2) \leq C h_1^2 \sum_{j=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2).$$

Combining (3.19) and the above estimate we have

$$\lambda_{\text{min}}(A) \leq \frac{a(u_h, u_h)}{\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2} \leq C \frac{a(u, u)}{\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2} \leq C h_1^2. \quad (3.21)$$

The right side of (3.9) follows now from (3.17) and (3.21).

**Remark 3.4** The SCS constant $\gamma$ in the above theorem depends on $r = \frac{h_2}{h_1}$. We have that $\gamma \not\to 1$ for $r \to 0$. A justification for this statement is that for $r \to 0$, the function in $\phi_1 V_1$ supported on $\Omega_1 \cap \Omega_2$ can be well approximated by functions in $\phi_2 V_2$ supported on $\Omega_2 \cap \Omega_1$. The above theorem still holds if the PU functions $\phi_1$ and $\phi_2$ are chosen to be piecewise polynomials of degree $n$. Numerical computations show that $\gamma$ increases to one as $h$ decreases and, on the good side, $\gamma$ decreases as $n$-the degree of PU functions increases.

**Remark 3.5** It is straightforward to apply the Schwartz’ alternating method to the variational problem (3.8). To obtain the solution in each step in the Schwartz’ alternating method, it is necessary to evaluate the following terms in the overlapping region

$$a(\phi_1 v_1, \phi_2 w_2) \quad v_i \in V_1, w_j \in V_2$$

which is not trivial in general case, because the topology of the intersection of the two meshes on the overlapping region has to be considered for computations.

## 4 Numerical Results

In this section, we carry out some numerical tests in one dimension. Consider the following Poisson equation

$$u'' = 1 \quad \text{on} \; (0, 1), \quad (4.22)$$

$$u(0) = u(1) = 0. \quad (4.23)$$

We consider two kinds of the overlapping region. The first kind is called thin overlapping because the width of the overlapping region is of the same order of a grid. To be precise, let $\Omega = (0, 1)$. Let $\Omega_1 = (0, 1/2)$ and divide it into $N$ intervals uniformly. The width of the interval is simply $h = \frac{1}{2N}$. The second region is given by $\Omega_2 = (1/2-h, 1)$ which is divided into $N+1$ intervals uniformly (see Fig. 2). Hence the overlapping region is $(1/2-h, 1/2)$. This is an example of thin overlapping region. The second kind is called fixed overlapping because the width of the overlapping is fixed and does not change when the grid is refined. To be precise, $\Omega_1 = (0, 1/2)$, $\Omega_2 = (1/4, 1/2)$ and the overlapping region is always $\Omega_o = (1/4, 1/2)$. We use linear finite element basis functions and choose nonlinear polynomials for PU functions. Note that a simple choice of $\phi_1 = 1 - x$ and $\phi_2 = x$ (scaled to $[0, 1]$) does not work since $\{\phi_1v, \phi_2w\}$ are linearly dependent.

![Explicative Mesh and PU Functions](image)

Figure 2: Explicative Mesh and PU Functions.

We choose four pairs of PU functions.

- $\phi_1^1 = 1 - x^2$, $\phi_2^1 = x^2$,
- $\phi_1^2 = 1 - x^3$, $\phi_2^2 = x^3$,
- $\phi_1^3 = 1 + 2x^3 - 3x^2$, $\phi_2^3 = -2x^3 + 3x^2$,
- $\phi_1^4 = 6x^2 - 7x + 1$, $\phi_2^4 = -4x^2 + 5x$.

### 4.1 Error in $L^2$ norm and $H^1$ semi-norm for thin overlapping

For convergence of the SA method for PUFEM, it is reasonable to look at the error in $L^2$ norm and $H^1$ semi-norm. We fix the PU functions $\phi_1^1$ and $\phi_2^1$ in this subsection. Note that the overlapping region is one element wide, i.e., if the interval $[0, 1]$ is divided into $M = 2N$
The errors are shown in Table 1. We obtain $O(h^2)$ in $L_2$ norm and $O(h)$ in $H^1$ semi-norm. The results are consistent with the theory in [8] (c.f. Definition 2.1, 2.2 and Theorem 2.1 there). The estimate in [8] predicts the $O(h^2)$ convergence $L_2$ norm. Since the overlapping region $I_o$ changes (same as $h$) as $N$ changes, our case is slightly different. Instead, for the PU function we choose, we have the following estimate

$$\|\nabla \phi_k\|_{L^\infty(Y)} \leq \frac{C_G}{\text{diam} \Omega_1} \frac{1}{h^2}$$

where $C_G$ is a constant. For the thin overlapping region we consider, taking into account $\text{diam} \Omega_1 = 1/h$, we will still have the $O(h)$ convergence in $H^1$ semi-norm.

**Remark 4.1** The overlapping region we consider is different than that considered in [8]. Theorem 2.1 in [8] is not optimal for our case. Thus, our numerical results show an improvement for the approximation result of Theorem 2.1 for the special case of thin overlapping.

Next we shall look at the convergence of the SA method. We fix the number of iteration with 50 and record the error for different $N$ in Table 2. Note that when $N$ increases, the width of the overlapping region $I_o$ decreases. When $N$ is small, the errors in Table 2 match those in Table 1 well. This indicates that the convergence of the SA method is better. As we increase $N$, the width of the overlapping region decreases and the convergence is worse. This is an indicator that the SCS constant is getting larger as the overlapping region is smaller.

### 4.2 Error in $L^2$ norm and $H^1$ semi-norm for fixed overlapping

In this section, we will consider the case when the overlapping region is fixed. For the same problem above, we have $L_o = [1/4, 1/2]$ fixed. In Table 3, we give the error in $L_2$ norm and $H^1$ semi-norm. The order of convergence is consistent with the results in [8].

Note that when $N = 4$, the case we consider here is the same as the one we consider in the previous subsection. The errors are identical which are not shown. Next we look at the convergence of the SA method. Again, we fix the number of iteration at 50 and record the error for different $N$ in Table 4. The errors do not become worse as we increase the number of elements as we expected since the overlapping region is fixed.

### 4.3 The SCS constant

Since the convergence rate of the SA method is decided by the the SCS constant, it would be plausible to look at the constants when the meshes are refined. Following [1], Let $u \in V = X \oplus Y$ with $X$ and $Y$ being PUFEM subspaces. If we partition the stiffness matrix into blocks

$$B = \begin{bmatrix} B_{XX} & B_{XY} \\ B_{YX} & B_{YY} \end{bmatrix},$$

then $\gamma^2$ is the maximum eigenvalue of the problem

$$B_{XY} B_{Y}^{-1} B_{XX} = \mu B_{XX}.$$

Table 5 shows the results for thin overlapping region and fixed overlapping region. Note that the error is in $H^1$ semi-norm.

One observation is that from Theorem 2.2 the SCS constant $\gamma \approx \|u - u_k\|/\|u - u_{k-1}\|$ for $k$ sufficiently large. Fig. 3 shows the relative errors of SA method for both thin and fixed overlapping regions. As the iteration number getting large, the relative errors approach the corresponding $\gamma$ in both cases. We also look at the relation between $\gamma$ and the PU functions. In Table 6, we calculate $\gamma$ for thin overlapping and fixed overlapping. We fix $N = 32$. For both cases, $\phi^c$’s give the smallest $\gamma$ and $\phi^d$’s give the largest $\gamma$. Finally we look the effect of the mesh

### Table 1: The error of the PUFEM.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_N|_{L^2}$</th>
<th>Ratio</th>
<th>$|e_N|_{H^1}$</th>
<th>Ratio</th>
</tr>
</thead>
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<td>8</td>
<td>1.334E-3</td>
<td></td>
<td>3.375E-2</td>
<td></td>
</tr>
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<td>3.453E-4</td>
<td></td>
<td>1.746E-2</td>
<td>1.932</td>
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<td>8.774E-5</td>
<td></td>
<td>8.879E-3</td>
<td>1.967</td>
</tr>
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<td>2.211E-5</td>
<td></td>
<td>4.475E-3</td>
<td>1.984</td>
</tr>
</tbody>
</table>

### Table 2: The error of the SA method for PUFEM.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_N|_{L^2}$</th>
<th>$|e_N|_{H^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.340E-3</td>
<td>3.375E-2</td>
</tr>
<tr>
<td>16</td>
<td>4.064E-4</td>
<td>1.747E-2</td>
</tr>
<tr>
<td>32</td>
<td>1.171E-3</td>
<td>9.893E-3</td>
</tr>
<tr>
<td>64</td>
<td>7.601E-3</td>
<td>2.781E-2</td>
</tr>
</tbody>
</table>

### Table 3: The error of the PUFEM for fixed overlapping region.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_N|_{L^2}$</th>
<th>Ratio</th>
<th>$|e_N|_{H^1}$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.235E-3</td>
<td></td>
<td>3.125E-2</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3.088E-4</td>
<td>4.000</td>
<td>1.563E-2</td>
<td>2.000</td>
</tr>
<tr>
<td>32</td>
<td>7.720E-5</td>
<td>4.000</td>
<td>7.813E-3</td>
<td>2.000</td>
</tr>
<tr>
<td>64</td>
<td>1.930E-5</td>
<td>4.000</td>
<td>3.906E-3</td>
<td>2.000</td>
</tr>
</tbody>
</table>

### Table 4: The error of the SA method for PUFEM for fixed overlapping region.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_N|_{L^2}$</th>
<th>$|e_N|_{H^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.318E-3</td>
<td>3.140E-2</td>
</tr>
<tr>
<td>16</td>
<td>5.032E-4</td>
<td>1.855E-2</td>
</tr>
<tr>
<td>32</td>
<td>1.611E-4</td>
<td>1.109E-2</td>
</tr>
<tr>
<td>64</td>
<td>7.008E-5</td>
<td>6.081E-3</td>
</tr>
</tbody>
</table>
Table 5: The CSC constants $\gamma^2$.

<table>
<thead>
<tr>
<th>N</th>
<th>fixedOverlapping</th>
<th>ThinOverlapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.8571</td>
<td>0.8571</td>
</tr>
<tr>
<td>8</td>
<td>0.9489</td>
<td>0.8658</td>
</tr>
<tr>
<td>16</td>
<td>0.9870</td>
<td>0.8869</td>
</tr>
<tr>
<td>32</td>
<td>0.9969</td>
<td>0.9239</td>
</tr>
<tr>
<td>64</td>
<td>0.9993</td>
<td>0.9572</td>
</tr>
</tbody>
</table>

Table 6: The SCS constants $\gamma^2$.

<table>
<thead>
<tr>
<th>N</th>
<th>$\phi_a$</th>
<th>$\phi_b$</th>
<th>$\phi_c$</th>
<th>$\phi_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.9239</td>
<td>0.9388</td>
<td>0.9005</td>
<td>0.9883</td>
</tr>
<tr>
<td>8</td>
<td>0.9674</td>
<td>0.9969</td>
<td>0.9990</td>
<td>0.9999</td>
</tr>
<tr>
<td>16</td>
<td>0.9969</td>
<td>0.9990</td>
<td>0.9764</td>
<td>0.9999</td>
</tr>
<tr>
<td>32</td>
<td>0.9969</td>
<td>0.9990</td>
<td>0.9764</td>
<td>0.9999</td>
</tr>
<tr>
<td>64</td>
<td>0.9990</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

Table 7: The SCS constants $\gamma^2$ for fixed overlapping.

<table>
<thead>
<tr>
<th>N</th>
<th>$\phi_a$</th>
<th>$\phi_b$</th>
<th>$\phi_c$</th>
<th>$\phi_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.8571</td>
<td>0.7995</td>
<td>0.5530</td>
<td>0.8813</td>
</tr>
<tr>
<td>8</td>
<td>0.9489</td>
<td>0.9504</td>
<td>0.7669</td>
<td>0.9888</td>
</tr>
<tr>
<td>16</td>
<td>0.9870</td>
<td>0.9924</td>
<td>0.9122</td>
<td>0.9986</td>
</tr>
<tr>
<td>32</td>
<td>0.9969</td>
<td>0.9990</td>
<td>0.9764</td>
<td>0.9998</td>
</tr>
<tr>
<td>64</td>
<td>0.9990</td>
<td>0.9999</td>
<td>0.9941</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

Table 8: The SCS constants $\gamma^2$ for thin overlapping.

<table>
<thead>
<tr>
<th>N</th>
<th>$\phi_a$</th>
<th>$\phi_b$</th>
<th>$\phi_c$</th>
<th>$\phi_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.8571</td>
<td>0.7995</td>
<td>0.5530</td>
<td>0.8813</td>
</tr>
<tr>
<td>8</td>
<td>0.8657</td>
<td>0.8398</td>
<td>0.6488</td>
<td>0.9502</td>
</tr>
<tr>
<td>16</td>
<td>0.8868</td>
<td>0.8936</td>
<td>0.8096</td>
<td>0.9763</td>
</tr>
<tr>
<td>32</td>
<td>0.9239</td>
<td>0.9388</td>
<td>0.9005</td>
<td>0.9883</td>
</tr>
<tr>
<td>64</td>
<td>0.9572</td>
<td>0.9674</td>
<td>0.9491</td>
<td>0.9942</td>
</tr>
</tbody>
</table>

Figure 3: Relative errors of the SA method ($N = 32$).

refinement on $\gamma$. Each time we cut the mesh size by half and calculate the SCS constant. For both thin and fixed overlapping, the constant is getting larger (closer to 1) as the mesh size decreases (see Table 7 and Table 8). Again, the PU functions $\phi_c$’s are the best in the sense that they give the smallest $\gamma$ among all the test PU functions at the same mesh refinement level.

5 Acknowledgements

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References


