GUIDELINES FOR THE METHOD OF UNDETERMINED COEFFICIENTS

Given a *constant coefficient* linear differential equation

$$ay'' + by' + cy = g(x),$$

where g is an exponential, a simple sinusoidal function, a polynomial, or a product of these functions:

1. Find a pair of linearly independent solutions of the homogeneous problem: $\{y_1, y_2\}$.
2. If g is **NOT** a solution of the homogeneous equation, take a trial solution of the same type as g as suggested in the table below:

Forcing Function	**Trial Solution**
ae^{rt}	Ae^{rt}
$a\sin(\omega t)$ or $a\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
at^n n a positive integer	$P(t)$ P a general polynomial of degree n
$at^n e^{rt}$ n a positive integer	$P(t)e^{rt}$ P a general polynomial of degree n
$t^n[a\sin(\omega t) + b\cos(\omega t)]$ n a positive integer	$P(t)[A\sin(\omega t) + B\cos(\omega t)]$ P a general polynomial of degree n
$e^{rt}[a\sin(\omega t) + b\cos(\omega t)]$	$e^{rt}[A\sin(\omega t) + B\cos(\omega t)]$

3. If g is a solution of the homogeneous problem, take a trial solution of the same type as g multiplied by the lowest power of t for which **NO TERM** of the trial solution is a solution of the homogeneous equation.
4. Substitute the trial solution into the differential equation and solve for the undetermined coefficients so that it is a particular solution y_p.
5. Set $y(t) = y_p(t) + [c_1 y_1(t) + c_2 y_2(t)]$ where the constants c_1 and c_2 can be determined if initial conditions are given.
6. If g is a sum of the type of forcing function described above, split the problem into simpler parts. Find a particular solution for each of these, then add particular solutions to obtain y_p for the complete equation.

Examples: In most of the following examples, we discuss the differential equation

$$y'' + y' - 6y = g(t),$$

for various choices of the forcing function g. Note that the associated homogeneous equation has characteristic polynomial $\lambda^2 + \lambda - 6$ which two distinct real roots, $\lambda_1 = -3$, and $\lambda_2 = 2$. Hence the two linearly independent solutions for the homogeneous equation are $y_1 = e^{-3t}$ and $y_2 = e^{2t}$.

(a) If $g(t) = 5e^{4t}$, then we take a trial solution $y_p(t) := Ae^{4t}$, with $y_p' = 4Ae^{4t}$ and $y_p'' = 16Ae^{4t}$. Then, we substitute these values into the differential equation. One convenient way to organize our work is to write things down as follows:

$$\begin{array}{rcl} y_p''(t) & = & 16Ae^{4t} \\ +y_p'(t) & = & 4e^{4t} \\ -6y_p(t) & = & -6Ae^{4t} \\ \hline \end{array}$$

$$\begin{array}{rcl} y_p'' + y_p' - 6y & = & 14Ae^{4t} \\ 14Ae^{4t} & = & 5e^{4t} \end{array}$$

or

Hence, $14A = 5$ or $A = 5/14$ and the particular solution for this problem is

$$y_p(t) = \frac{5}{14}e^{4t},$$

with general solution

$$y(t) = \frac{5}{14}e^{4t} + \left[c_1\, e^{-3t} + c_2\, e^{2t}\right].$$

Suppose that we are given initial conditions $y(0) = 1/2$ and $y'(0) = \sqrt{2}\pi/2$. Then, since

$$y'(t) = \frac{5}{14}e^{4t} + \left[-3c_1e^{-3t} + 2c_2e^{2t}\right],$$

we can compute the constants c_1, c_2 so that the solution satisfies the initial conditions. Indeed,

$$\begin{array}{rcl} y(0) & = & \frac{5}{14} + c_1 + c_2 = \frac{1}{2} \\ y'(0) & = & \frac{20}{14} - 3c_1 + 2c_2 = \frac{\sqrt{2}\pi}{2} \end{array}$$

or

$$\begin{aligned} c_1 + c_2 &= \frac{1}{7} \\ -3c_1 + 2c_2 &= \frac{7\sqrt{2}\pi - 20}{14} \end{aligned}$$

From the first, $c_1 = (1/7) - c_2$ so that, substituting into the second yields

$$-3\left(\frac{1}{7} - c_2\right) + 2c_2 = \frac{7\sqrt{2}\pi - 20}{14}$$

or

$$5c_2 = \frac{7\sqrt{2}\pi - 20}{14} + \frac{3}{7}$$

so that

$$c_2 = \frac{\sqrt{2}\pi - 2}{10}, \quad \text{and} \quad c_1 = \frac{1}{7}\frac{\sqrt{2}\pi - 2}{10}.$$

Thus we have the solution to the initial value problem:

$$y(t) = \frac{5}{14}e^{4t} + \left(\frac{12}{35} - \frac{\sqrt{2}\pi}{10}\right)e^{-3t} + \left(-\frac{1}{5} + \frac{\sqrt{2}\pi}{10}\right)e^{2t}.$$

(b) Let $g(t) = 4e^{2t}$, and take a trial solution $y_p(t) = Ae^{2t}$. Then $y_p'(t) = 2Ae^{2t}$ and $y_p''(t) = 4Ae^{2t}$. Then,

$$\begin{aligned} y_p''(t) &= 4Ae^{2t} \\ +y_p'(t) &= 2e^{2t} \\ -6y_p(t) &= -6Ae^{2t} \\ \hline y_p'' + y_p' - 6y &= 0Ae^{4t} \\ 0 &= 5e^{2t} \end{aligned}$$

or

From this result, we conclued that we have chosen the wrong trial function. The difficulty here is that Ae^{2t} is a solution of the homogeneous problem, so it cannot be used as a trial function because any constant multiple will also be a solution of the homogeneous problem.

The correct choice is $y_p(t) = A\,t\,e^{2t}$, with $y_p' = A[2\,t\,e^{2t} + e^{2t}]$ and $y_p'' = A[4\,t\,e^{2t} + 4e^{2t}]$. Then

$$
\begin{array}{rcl}
y_p''(t) & = & 4Ate^{2ta+4Ae^{2t}} \\
+y_p'(t) & = & 2Ate^{2t} + Ae^{2t} \\
-6y_p(t) & = & -6Ate^{2t} \\
\hline
y_p'' + y_p' - 6y & = & 5Ae^{2t}
\end{array}
$$

or

$$5Ae^{2t} = 4e^{4t},$$

so that $5A = 4$ or $A = 4/5$. It follows that a general solution is

$$y(t) = \frac{4}{5}te^{2t} + \left[c_1e^{-3t} + c_2e^{2t}\right].$$

(c) Suppose that $g(t) = 5t$. We take a trial solution in the form of a general polynomial of degree one, $y_p(t) = At+B$ with $y_p' = A$ and $y_p'' = 0$. Hence

$$
\begin{array}{rcl}
y_p''(t) & = & 0 \\
+y_p'(t) & = & A \\
-6y_p(t) & = & -6At - 6B \\
\hline
y_p'' + y_p' - 6y & = & -6At + (A - 6B)
\end{array}
$$

or

$$-6At + (A - 6B) = 5t,$$

It follows that $A - 6B = 0$ and $-6A = 5$, and therefore we find that $A = -(5/6), B = -(5/36)$, which results in the particular solution

$$y_p(t) = -\frac{5}{6}t - \frac{5}{36}.$$

CAUTIONARY NOTE: Suppose we had chosen the trial function simply as $y_p(t) = At$ with no constant term B which has the same derivatives as our first selection. Then substitution yields

$$y_p''(t) + y_p'(t) - 6y_p(t) = A - 6At = 5t.$$

This last equation, $A - 6At = 5t$ has no constant solution A and the method fails.

In order to illustrate the next point, we will look at a different differential equation:

$$y'' + y' + \frac{1}{4}y = te^{-\frac{t}{2}}.$$

(d) It is easy to check that the characteristic polynomial $\lambda^2 + \lambda + (1/4)$ has a double root $\lambda = -(1/2)$. While we may be tempted to take a trial solution of the form $y_p(t) = (A + Bt)e^{-\frac{t}{2}}$, we see that such a choice is doomed as it is simply a combination of the solutions to the homogeneous problem. We then try the form $y_p(t) = t(A + Bt)e^{-\frac{t}{2}} = (Bt^2 + At)e^{-\frac{t}{2}}$. Then a tedious calculation leads to:

$$\begin{aligned} y_p' &= -\frac{1}{2}(At + Bt^2)e^{-\frac{t}{2}} + (A + 2Bt)e^{-\frac{t}{2}} \\ &= \left[-\frac{1}{2}Bt^2 + (2B - \frac{1}{2}A)t) + A\right] e^{-\frac{t}{2}} \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \frac{1}{4}(At + Bt^2)e^{-\frac{t}{2}} + (A + 2Bt)e^{-\frac{t}{2}} + 2Be^{-\frac{t}{2}} \\ &= \left[\frac{1}{4}Bt^2 + (-2B + \frac{1}{4}A)t) + (2B - A)\right] e^{-\frac{t}{2}} \end{aligned}$$

Using these results, we have:

$$\begin{array}{rcc} y_p''(t) & = & \left[\frac{1}{4}Bt^2 + (-2B + \frac{1}{4}A)t) + (2B - A)\right] e^{-\frac{t}{2}} \\ +y_p'(t) & = & \left[-\frac{1}{2}Bt^2 + (2B - \frac{1}{2}A)t) + A\right] e^{-\frac{t}{2}} \\ \frac{1}{4}y_p(t) & = & \left(\frac{1}{4}Bt^2 + \frac{1}{4}At\right) e^{-\frac{t}{2}} \\ \hline y_p'' + y_p' - 6y & = & 0t^2 + 0t + 2Be^{\frac{t}{2}} \\ & & \text{or} \\ 2Be^{-\frac{t}{2}} & = & te^{-\frac{t}{2}}, \end{array}$$

Again, the resulting equation $2B = t$ has no constant solution. In this case our error is that, while we have avoided a solution of the homogeneous equation by multiplying the polynomial by t, we still have that ONE of the terms in the new trial solution is a solution of the homogeneous equation, namely the term $A\,t\,e^{-\frac{t}{2}}$. We must multiply by a high enough power of t so that NO PART of the resulting trial solution is a solution of the homogeneous problem. In this case, we need a trial solution of the form $y_p(t) = t^2(A + Bt)e^{-\frac{t}{2}}$.

(e) Now we return to the equation $y'' + y' - 6y = g(t)$ and consider the case when $g(t) = 5t + 4e^{2t}$. Here we split the problem into two parts, which we first solve separately and then add to get the partucular solution of the given non-homogeneous equation. So we consider separately (e1) $y'' + y' - 6y = 5t$

(e2) $y'' + y' - 6y = 4e^{2t}$.

Since this example is meant only to illustrate the point, we have chosen two problems which we have already solved above in parts (c) and (b) respectively. Thus a particular solution to (e1) is $y_p(t) = -(5/6)t + (5/36)$ while a solution of (e2) is $y_p(t) = (4/5)\, t\, e^{2t}$. Hence a particular solution of the full problem corresponding to the forcing function $g(t) = 5t + 4e^{2t}$ is

$$y_p(t) = -\frac{5}{6}t - \frac{5}{36} + \frac{4}{5}te^{2t}.$$