GUIDELINES FOR CONSTANT COEFFICIENT LINEAR SYSTEMS

Given the **constant coefficient** linear differential system

$$\dot{\mathbf{x}} = A\,\mathbf{x},$$

where A is an $n \times n$ matrix of constants, one may find linearly independent solutions as follows:

1. Compute the matrix $A - \lambda I$ and the characteristic equation $\det(A - \lambda I) = 0$.
2. Find all the roots of the characteristic equation. The procedure to follow depends on the nature of the roots (eigenvalues) of the characteristic equation.

The rest of the procedure we describe the procedure for 2×2 systems. The procedure is the same for systems of any size.

3. If the roots are **real and distinct** then
   (a) For each eigenvalue λ_1 and λ_2, find a corresponding eigenvector $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ by solving the equation

   $$(A - \lambda_i I)\mathbf{v}^{(i)} = 0, \quad i = 1, 2.$$

   (b) Form the general solution

   $$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t),$$

   where $\mathbf{x}^{(i)}(t) := \mathbf{v}^{(i)}e^{\lambda_i t}, i = 1, 2$.

   (c) If initial data are given, $\mathbf{x}(t_0) = \mathbf{x}_0$, then find the appropriate values of the constants by solving the system

   $$\mathbf{x}_0 = c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0)$$

   for the unknowns c_1 and c_2.

4. If the roots are **real and equal** then

(a) Using the single real eigenvalue λ, find an eigenvector $\mathbf{v} \neq 0$ and a **generalized eigenvector** $\mathbf{u} \neq 0$ by solving the equations:

$$(A - \lambda I)\mathbf{v} = 0$$

and

$$(A - \lambda I)\mathbf{u} = \mathbf{v}.$$

(b) Form the general solution

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t),$$

where $\mathbf{x}^{(1)}(t) = \mathbf{v}e^{\lambda t}$ and $\mathbf{x}^{(2)}(t) = (\mathbf{u} + \mathbf{v}t)e^{\lambda t}$.

(c) If initial data are given, $\mathbf{x}(t_0) = \mathbf{x}_0$, then find the appropriate values of the constants by solving the system

$$\mathbf{x}_0 = c_1\mathbf{x}^{(1)}(t_0) + \mathbf{x}^{(2)}(t_0)$$

for the unknowns c_1 and c_2.

5. If the roots are **complex conjugates** then

(a) Using one of the complex eigenvalues λ find the corresponding (complex) eigenvector $\mathbf{w} = \mathbf{w}_1 + i\mathbf{w}_2$ by solving the algebraic equation

$$(A - \lambda I)\mathbf{w} = 0.$$

(b) Form two linearly independent *real* solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ as follows

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} u_1\cos(qt) - v_1\sin(qt) \\ u_2\cos(qt) - v_2\sin(qt) \end{pmatrix} \exp(pt),$$

and

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} u_1\sin(qt) + v_1\cos(qt) \\ u_2\sin(qt) + v_2\cos(qt) \end{pmatrix} \exp(pt),$$

where $\lambda = p + iq, w_1 = u_1 + iv_1$, and $w_2 = u_2 + iv_2$. The general *real* solution is then given by

$$\mathbf{x}_1(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t).$$

(c) If initial data $\mathbf{x}(t_0) = \mathbf{x}_0$ are given, find c_1 and c_2 from

$$\mathbf{x}_0 = c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0).$$