

ALGORITHM FOR SOLVING
 $dx/dt = Ax$
CASE OF REPEATED REAL
EIGENVALUES

1. Determine the eigenvalue λ which is a root of the characteristic equation

$$\det (A - \lambda I) = 0.$$

2. Find two corresponding linearly independent eigenvectors \mathbf{b}_1 and \mathbf{b}_2 which are both solutions of

$$(A - \lambda I)\mathbf{b} = 0,$$

if possible. If this is not possible, then find a single eigenvector and then compute a solution of the **INHOMOGENEOUS** equation

$$(A - \lambda I)\mathbf{k} = \mathbf{b}_1.$$

Such a vector \mathbf{k} is called a **generalized eigenvector**.

3. In the first case, form the two linearly independent solutions using the usual rule

$$\mathbf{x}^{(1)}(t) = \mathbf{b}_1 e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \mathbf{b}_2 e^{\lambda t}.$$

The general solution is then

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t).$$

In the case that there is only one linearly independent eigenvector, \mathbf{b}_1 there are then two solutions of the forms

$$\mathbf{x}^{(1)} = \mathbf{b}_1 e^{\lambda t},$$

and

$$\mathbf{x}^{(2)} = \mathbf{b}_1 t e^{\lambda t} + \mathbf{k} e^{\lambda t}.$$

4. If initial data $\mathbf{x}(t_0) = \mathbf{x}^0$ are given, find the constants c_1 and c_2 by solving the *algebraic* system

$$\mathbf{x}^0 = c_1 \mathbf{x}^{(1)}(t_0) + c_2 \mathbf{x}^{(2)}(t_0).$$

EXAMPLE 1 :

Find the solution of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}.$$

The eigenvalues are the roots of

$$\begin{vmatrix} 1 - \lambda & -2 & 2 \\ -2 & 1 - \lambda & -2 \\ 2 & -2 & 1 - \lambda \end{vmatrix} = -(\lambda + 1)^2(\lambda - 5) = 0.$$

So $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$.

Using the first eigenvalue, we form the system

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to the single equation $x - y + z = 0$ which is one equation in three unknowns. Therefore we have two arbitrary choices to make. If we choose $y = 1$ and $z = 0$ then we obtain $x = 1$ so that one eigenvector associated with $\lambda = -1$ is

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

On the other hand, the choice of $x = 0$ and $y = 1$ leads to $z = 1$ and hence to the eigenvector

$$\mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

which is linearly independent from \mathbf{b}_1 since neither vector is a constant multiple of the other. Thus in this case, the repeated eigenvalue leads to two linearly independent eigenvectors and thus to two linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}, \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}.$$

EXERCISE :

Show that an eigenvector associated with the third eigenvalue $\lambda = 5$ is given by

$$\mathbf{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t},$$

and find the form of the general solution.

EXAMPLE 2 :

Find the solution of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{x}.$$

The eigenvalues are the roots of

$$\begin{vmatrix} 3 - \lambda & -18 \\ 2 & -9 - \lambda \end{vmatrix} = (\lambda + 3)^2 = 0,$$

so $\lambda_1 = \lambda_2 = -3$. The vector \mathbf{b} , found by solving the system of equations

$$6b_1 - 18b_2 = 0, \quad 2b_1 - 6b_2 = 0,$$

is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

so that one solution is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{\lambda t}.$$

Now, to find a second linearly independent solution, we must find the generalized eigenvector \mathbf{k} by solving the inhomogeneous system

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

or, equivalently, the pair of equations

$$\begin{aligned} 6k_1 - 18k_2 &= 3, \\ 2k_1 - 6k_2 &= 1. \end{aligned} \tag{1}$$

Since this system is obviously equivalent to one equation, we have an infinite number of choices for k_1 and k_2 . For simplicity, choose $k_2 = 0$ so that $k_1 = 1/2$. Hence

$$\mathbf{k} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

The second linearly independent solution of the differential equation is then

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}.$$

and the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right].$$

EXERCISE :

Find the general solutions of

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$