ALGORITHM FOR SOLVING $dx/dt = Ax$ CASE OF COMPLEX CONJUGATE EIGENVALUES

1. Determine the eigenvalues λ and λ^* which are roots of the characteristic equation
$$\det\,(A - \lambda I) = 0.$$

2. Find the corresponding eigenvectors $\mathbf{b}$ and $\mathbf{b}^*$ which are solutions of
$$(A - \lambda I)\mathbf{b} = 0, \;\; (A - \lambda^* I)\mathbf{b}^* = 0.$$

3. Form the **REAL** vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ according to the definitions:
$$\mathbf{k}_1 = \frac{1}{2}[\mathbf{b} + \mathbf{b}^*], \;\; \mathbf{k}_2 = \frac{i}{2}[-\mathbf{b} + \mathbf{b}^*]$$

4. Two linearly independent **REAL** solutions are given by
$$\mathbf{x}^{(1)}(t) = (\mathbf{k}_1\ \cos\ \beta t - \mathbf{k}_2\ \sin\ \beta t)e^{\alpha t},$$
and
$$\mathbf{x}^{(2)}(t) = (\mathbf{k}_2\ \cos\ \beta t + \mathbf{k}_1\ \sin\ \beta t)e^{\alpha t},$$
where $\lambda = \alpha + i\,\beta$. The general solution is
$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t).$$

5. If initial data $\mathbf{x}(t_0) = \mathbf{x}^0$ are given, find the constants c_1 and c_2 by solving the *algebraic* system
$$\mathbf{x}^0 = c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0).$$

EXAMPLE :

Find the solution of the system
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}\mathbf{x}.$$

The eigenvalues are the roots of

$$\begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0\,,$$

so $\lambda = 2i\,,\ \lambda^* = -2i$. The vector $\mathbf{b}$, found by solving the system of equations

$$[2-2i]\,b_1 + 8b_2 = 0\,, \quad -b_1 + [-2-2i]\,b_2 = 0\,,$$

is

$$\begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Now, from step (3) above,

$$\mathbf{k}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

and

$$\mathbf{k}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Since λ is pure imaginary (and so $\alpha = 0$), the general solution of the system is

$$\mathbf{x}(t) = c_1 \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos\,2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin\,2t \right] + c_2 \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos\,2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin\,2t \right]$$

or

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\cos\,2t - 2\sin\,2t \\ -\cos\,2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos\,2t + 2\sin\,2t \\ -\sin\,2t \end{pmatrix}.$$

1. Find the general solutions of

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{x}\,.$$

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2. Find the solution of the initial value problem

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$