ALGORITHM FOR SOLVING
$dx/dt = Ax$
DISTINCT REAL EIGENVALUE CASE

1. Determine the eigenvalues λ_1 and λ_2 which are roots of the characteristic equation
$$\det\,(A - \lambda I) = 0.$$

2. Find the corresponding eigenvectors $\mathbf{b}^1$ and $\mathbf{b}^2$ which are solutions of
$$(A - \lambda_j I)\mathbf{b} = 0,\ j = 1, 2.$$

3. Two linearly independent solutions are given by
$$\mathbf{x}^{(1)}(t) = \mathbf{b}^1 e^{\lambda_1\, t}, \quad \mathbf{x}^{(2)}(t) = \mathbf{b}^2 e^{\lambda_2\, t},$$
and the general solution is
$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t).$$

4. If initial data $\mathbf{x}(t_0) = \mathbf{x}^{(0)}$ are given, find the constants c_1 and c_2 by solving the *algebraic* system
$$\mathbf{x}^{(0)} = c_1 \mathbf{x}^{(1)}(t_0) + c_2 \mathbf{x}^{(2)}(t_0).$$

EXAMPLE :

Find the solution of the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -5 & 4 \\ 1 & -2 \end{bmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

The eigenvalues are the roots of

$$\begin{vmatrix} -5-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0\,,$$

so $\lambda_1 = -1\,,\ \lambda_2 = -6$. A vector $\mathbf{b}^1$, found by solving the system of equations

$$[-5-(-1)]\,b_1 + 4b_2 = 0\,, \quad b_1 + [-2-(-1)]\,b_2 = 0\,,$$

is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly,

$$\mathbf{b}^2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

satisfies

$$[-5-(-6)]\, b_1 + 4b_2 = 0\,, \quad b_1 + [-2-(-6)]\, b_2 = 0\,.$$

The corresponding solutions are

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\,, \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{-6t}\,,$$

and the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{-6t}\,.$$

To obtain the solution satisfying the initial conditions, c_1 and c_2 must satisfy

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

or

$$3 = c_1 + 4c_2\,, \quad -2 = c_1 - c_2\,.$$

Solving these gives $c_1 = -1$, $c_2 = 1$, and the desired solution is

$$\begin{aligned} \mathbf{x}(t) &= (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + (1) \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{-6t} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{-6t}\,. \end{aligned}$$

1. Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \mathbf{x} .$$

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2. Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix} \mathbf{x} .$$