GUIDELINES FOR SEPARABLE FIRST ORDER EQUATIONS

Given the *separable* differential equation

$$y' = f(x)\, g(y),$$

the steps to find a solution are the following:

Step 1: Find all constant (equilibrium) solutions by solving the equation $g(y) = 0$.

Step 2: Separate variables and integrate to find an **implicit solution** with possible restrictions on the additive constant.

Step 3: If possible, find **explicit solutions** from the implicit solution. Determine their intervals of definition.

Step 4: If initial data are given, use them in Step 2 to determine the additive constant and again in Step 3 to insure that you have the desired explicit solution.

Example: Consider the differential equation

$$\frac{dy}{dx} = \frac{x - x\,y^2}{y + x^2\,y}.$$

1. $\dfrac{x - x\,y^2}{y + x^2\,y} = 0$ if and only if

   $$x\,(1 - y^2) = x\,(1 - y)\,(1 + y) = 0$$

   so the equilibrium solutions are

   $$y(x) \equiv 1, \quad \text{and} \quad y(x) \equiv -1.$$

2. Separating variables in the equation we get

   $$\left[\frac{y}{(1-y)\,(1+y)}\right] \left(\frac{dy}{dx}\right) = \frac{x}{1+x^2}$$

   or

   $$\frac{1}{2}\left[\frac{1}{1-y} - \frac{1}{1+y}\right] \left(\frac{dy}{dx}\right) = \frac{x}{1+x^2}.$$

   Integrating both sides we get

   $$-\ln|y-1| - \ln|y+1| = \ln(1+x^2) + C$$

or

$$\frac{1}{|1-y^2|} = K(1+x^2)$$

Which already gives the *implicit* form of the solution of the differential equation. However, we can make some simplification which, after a little algebra yields:

$$y^2(x) = \frac{x^2+C}{1+x^2}. \qquad \textbf{CHECK!}$$

Note that the constant C must be positive so that we can write $C = a^2$.

3. Using this last form, we can easily find two explicit solutions:

$$y_1(x) = \sqrt{\frac{x^2+a^2}{1+x^2}}, \text{ and } y_2(x) = -\sqrt{\frac{x^2+a^2}{1+x^2}}.$$

In each case the requirement that the radicand be non–negative is satisfied. This means that each solution is defined for *all* values of x.

4. If we are given initial data, in this case data of the form $y(0) = y_0$ then we can determine the constant of integration even with the form

$$\ln\left(\frac{1}{|1-y^2|}\right) = \ln(1+x^2) + C$$

for, applying the initial condition, we have

$$\ln\left(\frac{1}{|1-y_0^2|}\right) = C$$

and this gives an explicit value for C. For example, if $y(0) = 3$ then $C = \ln\frac{1}{8}$. We then have

$$\frac{1}{|1-y^2|} = \frac{1}{8}(1+x^2).$$

Again, a little algebra then yields,

$$y^2(x) = \frac{x^2+9}{1+x^2}.$$

<u>CAUTION</u>: YOU MUST BE CAREFUL HERE! The algebra leads to $y^2 = (x^2+1\pm 8)/(1+x^2)$ and we CANNOT choose -8 because then the constant would be negative!

For the given initial condition, $y_0 = 3 > 0$, the required solution is obviously $y = y_1(x)$.