1. \[ \frac{dy}{dx} = 3x^2e^{-y}, \quad y(0) = 1. \]

Since \( f(x,y) = 3x^2e^{-y} \) and \( e^{-y} \neq 0 \) for any \( y \), there are no equilibrium solutions. Separating variables, we have

\[ e^{y(x)} \left( \frac{dy(x)}{dx} \right) = 3x^2 \]

and integrating each side with respect to the independent variable, we have

\[ \int^x e^{y(s)} \left( \frac{dy(s)}{ds} \right) ds = \int^x 3s^2 ds \]

or

\[ e^{y(x)} = x^3 + C, \]

which gives the solution \( y = y(x) \) in implicit form. The arbitrary constant of integration can be determined at this stage by invoking the initial condition \( y(0) = 1 \). This yields

\[ e^1 = 0 + C, \quad \text{or} \quad C = e \quad \text{giving} \quad e^y = e^{y(x)} = x^3 + e. \]

This last equation can be solved for \( y(x) \) to give an explicit form of the solution

\[ \ln e^{y(x)} = \ln (x^3 + e), \quad \text{or} \quad y(x) = \ln (x^3 + e). \]

Note that there are no absolute value signs occurring in the argument of the logarithmic function since the initial condition is positive. **However** this function is defined only for positive arguments hence the given solution is valid only on the semi-infinite interval \( x > -e^1. \)

2. \[ \frac{dy}{dx} = -\frac{y^2}{x}, \quad x \neq 0. \]

There is a constant solution \( y(x) \equiv 0 \) which is defined on any interval which does not contain \( x = 0 \) since the differential equation is not defined at \( x = 0 \).

Separating variables we have

\[ \left( \frac{1}{(y(x))^2} \right) \frac{dy(x)}{dx} = -\frac{1}{x} \]

and integrating each side with respect to the independent variable, we have

\[ \int^x \left( \frac{1}{(y(s))^2} \right) \left( \frac{dy(s)}{ds} \right) ds = -\int^x \left( \frac{1}{x} \right) ds \]
or

\[- \frac{1}{y(x)} = \ln |x| + C\]

which, again, gives the solution \( y = y(x) \) in implicit form. Solving for \( y \) yields the explicit solution

\[ y(x) = \frac{1}{\ln |x| + C}. \]

Notice that the existence and uniqueness theorem implies that any non-constant solution is either always positive or always negative. (Why?)

Moreover, recognizing that \( y(x) \) is defined only for an interval containing the initial condition, if the initial condition is, for example, \( y(1) = 2 \) why is the correct solution \( y(x) = \frac{1}{\ln x + \frac{1}{2}} \), for \( x > e^{\frac{1}{2}} \)? In particular, why the lower bound for \( x \)?

3.

\[ \frac{dy}{dx} = \frac{x^4}{y^4}. \]

In this example there are no constant solutions. Separating variables we have

\[ (y(x))^4 \left( \frac{dy(x)}{dx} \right) = x^4 \]

and integrating each side with respect to the independent variable, we have

\[ \int^x ((y(s))^4) \left( \frac{dy(s)}{ds} \right) ds = \int^x s^4 ds \]

or

\[ \frac{1}{5} (y(x))^5 = \frac{x^5}{5} + C \]

so that the implicit form of the solution is \( (y(x))^5 = x^5 + K \) where the constant \( C \) has been replaced by another arbitrary constant \( K = 5C \). What is the interval of definition? Note that \( y(x) = (x^5 + K)^{\frac{1}{5}} \) is defined for all \( x \in \mathbb{R} \) but is not differentiable at \( x = \alpha \) where \( \alpha^5 + K = 0 \). (Why not?)

Therefore, there is one solution defined on the semi-infinite interval \( \alpha < x < \infty \) and a second on the semi-infinite interval \( -\infty < x < \alpha \). However, both have the explicit form

\[ y(x) = (x^5 + K)^{\frac{1}{5}}. \]
4. \[ \frac{dy}{dx} = \cos \frac{x}{y}. \]

Since there are no equilibrium solutions we simply separate variables to obtain:

\[ y(x) \left( \frac{dy(x)}{dx} \right) = \cos x, \]

and integrating each side with respect to the independent variable, we have

\[ \int^x (y(s)) \left( \frac{dy(s)}{ds} \right) ds = \int^x \cos s ds \]

or

\[ \frac{1}{2} (y(x))^2 = \sin x + C \]

so that the implicit form of the solution is \((y(x))^2 = 2 \sin x + C\). Note that the differential equation is not defined at \(y = 0\) and that, since \(\frac{1}{2}y^2 > 0\), the constant \(C\) must satisfy the inequality \(\sin x + C > 0\). Since \(-1 \leq \sin x \leq 1\), we must have \(C > -1\). There are therefore two possible explicit solutions \(y_1 = \sqrt{2} \sin x + C\) and \(y_2 = -\sqrt{2} \sin x + C\). Which is the appropriate one depends on whether the given initial condition is positive or negative.

5. \[ \frac{dy}{dx} = \frac{x^2}{1 + y^5}. \]

Since there are no equilibrium solutions, we again simply separate variables and find:

\[ (1 + (y(x))^5) \left( \frac{dy(x)}{dx} \right) = x^2, \]

and integrating each side with respect to the independent variable, we have

\[ \int^x (1 + (y(s))^5) \left( \frac{dy(s)}{ds} \right) ds = \int^x s^2 ds \]

or

\[ \frac{1}{6} (y(x))^6 + y(x) = \frac{1}{3} x^3 + C \]

so that an implicit form of the solution is \((y(x))^6 + 6y(x) = 2x^3 + C\). This expression cannot be solved explicitly for \(y\). Try using the MAPLE command

\[ \text{dsolve} \ (diff(y(x), x) = (x^2)/(1 + y(x)^5), y(x)); \]

Nevertheless, it is possible to use a given initial condition to determine the appropriate value of \(C\). For example, if the initial condition is given as \(y(1) = 1\) then substitution of this condition in the general solution yields

\[ \frac{1}{6} \cdot 1^6 + 1 = \frac{1}{3} \cdot 1 + C, \quad \text{or} \quad C = \frac{5}{6}. \]
6. \[
\frac{dy}{dx} = -y(1 - y), \quad y(0) = 2.
\]

Here there are equilibrium solutions corresponding to the roots of the polynomial equation \(y(1 - y) = 0\). So the equilibrium solutions are \(y(x) \equiv 0\) and \(y(x) \equiv 1\).

Assuming that we are not at an equilibrium solution, we can separate variables as usual.

\[
\left( \frac{1}{y(x)(1 - y(x))} \right) \frac{dy(x)}{dx} = -1.
\]

In order to integrate the left hand side we must first make a decomposition into partial fractions. This results in the integration problem

\[
\int_{x}^{x} \left( \frac{1}{y(s)} \right) \left( \frac{dy(s)}{ds} \right) ds + \int_{x}^{x} \left( \frac{1}{1 - y(s)} \right) \left( \frac{dy(s)}{ds} \right) ds = -\int_{x}^{x} ds.
\]

Carrying out the computation yields:

\[
\ln |y(x)| - \ln |1 - y(x)| = -x + C, \quad \text{or equivalently} \quad \ln \left| \frac{y(x)}{1 - y(x)} \right| = -x + C.
\]

To solve the initial value problem, set \(x = 0\) and \(y = 2\):

\[
\ln \left| \frac{2}{1 - 2} \right| = \ln 2 = 0 + C, \quad \text{or} \quad C = \ln 2.
\]

Therefore, the implicit solution is given by

\[
\ln \left| \frac{y(x)}{1 - y(x)} \right| = -x + 2.
\]

Since \(y(0) = 2, (1 - y(x)) < 0 \) (why?), so that the implicit solution, after taking exponentials of both sides, is

\[
y(x) = 2 e^{-x}. \]

This last expression can be solved for \(y\) to find the explicit solution:

\[
y(x) = \frac{-2e^{-x}}{1 - 2e^{-x}} = \frac{2}{2 - e^{x}}.
\]

Note that the solution is valid for all \(x > \ln \frac{1}{2}\) and that \(y(x) > 1\) for all \(x \geq 0\).