Seperable Equations Examples–ANSWERS

1. $$\frac{dy}{dx} = 3x^2 e^{-y}\,, \quad y(0) = 1\,.$$

Since $f(x, y) = 3x^2e^{-y}$ and $e^{-y} \neq 0$ for any y, there are no equilibrium solutions. Separating variables, we have

$$e^{y(x)}\left(\frac{dy(x)}{dx}\right) = 3x^2$$

and integrating each side with respect to the independent variable, we have

$$\int^x e^{y(s)}\left(\frac{dy(s)}{ds}\right) ds = \int^x 3s^2\, ds$$

or

$$e^{y(x)} = x^3 + C,$$

which gives the solution $y = y(x)$ in implicit form. The arbitrary constant of integration can be determined at this stage by invoking the initial condition $y(0) = 1$. This yields

$$e^1 = 0 + C, \text{ or } C = e \text{ giving } e^y = e^{y(x)} = x^3 + e.$$

This last equation can be solved for $y(x)$ to give an explicit form of the solution

$$\ln e^{y(x)} = \ln(x^3 + e), \text{ or } y(x) = \ln(x^3 + e).$$

Note that there are no absolute value signs occuring in the argument of the logarithmic function since the initial condition is positive. **However** this function is defined *only* for positive arguments hence the given solution is valid only on the semi-infinite interval $x > -e^{\frac{1}{3}}$.

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2. $$\frac{dy}{dx} = -\frac{y^2}{x}\,, \quad x \neq 0\,.$$

There is a constant solution $y(x) \equiv 0$ which is defined on any interval which does not contain $x = 0$ since the differential equation is not defined at $x = 0$.

Separating variables we have

$$\left(\frac{1}{(y(x))^2}\right)\frac{dy(x)}{dx} = -\frac{1}{x}$$

and integrating each side with respect to the independent variable, we have

$$\int^x \left(\frac{1}{(y(s))^2}\right)\left(\frac{dy(s)}{ds}\right) ds = -\int^x \left(\frac{1}{x}\right) ds$$

or

$$-\frac{1}{y(x)} = \ln|x| + C$$

which, again, gives the solution $y = y(x)$ in implicit form. Solving for y yields the explicit solution

$$y(x) = \frac{1}{\ln|x| + C}.$$

Notice that the existence and uniqueness theorem implies that any non-constant solution is either always positive or always negative. (Why?)

Moreover, recognizing that $y(x)$ is defined only for an interval containing the initial condition, if the initial condition is, for example, $y(1) = 2$ why is the correct solution $y(x) = \dfrac{1}{\ln x + \frac{1}{2}}$, for $x > e^{\frac{1}{2}}$? In particular, why the lower bound for x?

3.
$$\frac{dy}{dx} = \frac{x^4}{y^4}.$$

In this example there are no constant solutions. Separating variables we have

$$(y(x))^4 \left(\frac{dy(x)}{dx}\right) = x^4$$

and integrating each side with respect to the independent variable, we have

$$\int^x \left((y(s))^4\right) \left(\frac{dy(s)}{ds}\right) ds = \int^x s^4\, ds$$

or

$$\frac{1}{5}(y(x))^5 = \frac{x^5}{5} + C$$

so that the implicit form of the solution is $(y(x))^5 = x^5 + K$ where the constant C has been replaced by another arbitrary constant $K = 5\,C$. What is the interval of definition? Note that $y(x) = (x^5 + K)^{\frac{1}{5}}$ is defined for all $x \in \mathbb{R}$ **but** is not *differentiable* at $x = \alpha$ where $\alpha^5 + K = 0$. (Why not?)

Therefore, there is one solution defined on the semi-infinite interval $\alpha < x < \infty$ and a second on the semi-infinite interval $-\infty < x < \alpha$. However, *both* have the explicit form

$$y(x) = (x^5 + K)^{\frac{1}{5}}.$$

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4. $$\frac{dy}{dx} = \frac{\cos\, x}{y}\,.$$

Since there are no equilibrium solutions we simiply separate variables to obtain:

$$y(x)\left(\frac{dy(x)}{dx}\right) \;=\; \cos x,$$

and integrating each side with respect to the independent variable, we have

$$\int^x (y(s))\left(\frac{dy(s)}{ds}\right)\, ds \;=\; \int^x \cos s\, ds$$

or

$$\frac{1}{2}\,(y(x))^2 \;=\; \sin x + C$$

so that the implicit form of the solution is $(y(x))^2 \;=\; \sin x + C$. Note that the differential equation is not defined at $y = 0$ and that, since $\frac{1}{2}y^2 > 0$, the constant C must satisfy the inequality $\sin x + C > 0$. Since $-1 \le \sin x \le 1$, we must have $C > -1$. There are therefore two possible explicit solutions $y_1 \;=\; \sqrt{2\sin x + C}$ and $y_2 \;=\; \sqrt{2\sin x + C}$. Which is the appropriate on depends on whether the given initial condition is positive or negative.

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5. $$\frac{dy}{dx} = \frac{x^2}{1+y^5}\,.$$

Since there are no equilibrium solutions, we again simply separate variables and find:

$$\left(1 + (y(x)^5\right)\left(\frac{dy(x)}{dx}\right) \;=\; x^2,$$

and integrating each side with respect to the independent variable, we have

$$\int^x (1 + (y(s))^5)\left(\frac{dy(s)}{ds}\right)\, ds \;=\; \int^x s^2\, ds$$

or

$$\frac{1}{6}\,(y(x))^6 + y(x) \;=\; \frac{1}{3}x^3 + C$$

so that an implicit form of the solution is $(y(x))^6 + 6y(x) \;=\; 2x^3 + C$. This expression cannot be solved explicitly for y. Try using the MAPLE command

dsolve $(diff(y(x), x) = (x^2)/(1 + y(x)^5), y(x));$

Nevertheless, it is possible to use a given initial condition to determine the appropriate value of C. For example, if the initial condition is given as $y(1) = 1$ then substitution of this condition in the general solution yields

$$\frac{1}{6}\,1^6 + 1 \;=\; \frac{1}{3}\,1 + C\,, \;\text{ or }\; C \;=\; \frac{5}{6}.$$

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6. $$\frac{dy}{dx} = -y(1-y)\,, \quad y(0) = 2\,.$$

Here there are equilibrium solutions corresponding to the roots of the polynomial equation $y(1-y) = 0$. So the equilibrium solutions are $y(x) \equiv 0$ and $y(x) \equiv 1$.

Assuming that we are not at an equilibrium solution, we can separate variables as usual.

$$\left(\frac{1}{y(x)(1+y(x))}\right)\frac{dy(x)}{dx} = -1.$$

In order to integrate the left hand side we must first make a decomposition into partial fractions. This results in the integration problem

$$\int^x \left(\frac{1}{y(s)}\right)\left(\frac{dy(s)}{ds}\right)\,ds + \int^x \left(\frac{1}{1+y(s)}\right)\left(\frac{dy(s)}{ds}\right)\,ds = -\int^x ds.$$

Carrying out the computation yields:

$$\ln|y(x)| - \ln|1+y(x)| = -x + C, \text{ or equivalently } \ln\left|\frac{y(s)}{1+y(s)}\right| = -x + C.$$

To solve the initial value problem, set $x = 0$ and $y = 2$:

$$\ln\left|\frac{2}{1-2}\right| = \ln 2 = 0 + C, \text{ or } C = \ln 2.$$

Therefore, the implicit solution is given by

$$\ln\left|\frac{y(s)}{1+y(s)}\right| = -x + \ln 2.$$

Since $y(0) = 2, (1 - y(x)) < 0$ (why?), so that the implicit solution, after taking exponentials of both sides, is

$$\frac{y(x)}{1-y(x)} = 2\,e^x.$$

This last expression can be solved for y to find the explicit solution:

$$y(x) = \frac{-2e^x}{1-2e^x} = \frac{2}{2-e^{-x}}.$$

Note that the solution is valid for all $x > \ln\frac{1}{2}$ and that $y(x) > 1$ for all $x \geq 0$.