EXAMPLES FOR SOLVING INITIAL VALUE PROBLEMS

$$a\ddot{x} + b\dot{x} + cx = 0$$
$$x(t_0) = x_0, \;\; \dot{x}(t_0) = \dot{x}_0.$$

Ich kam unerwartet auf meine Lösung und hatte vorher keine Ahnung, daß die Lösung einer algebraischen Gleichung in dieser Sache so nütlich sein könnte.

I came to my solution unexpectedly having had, beforehand, no idea that the solution of an algebraic equation could be so useful in this case.

— Leonhard Euler

In order to solve the second order linear initial value problem in the case of constant coeffients, we always follow the same steps to first find exponential solutions.

1. Write down the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

2. Find the roots of the characteristic equation. The nature of the roots is determined by the behavior of the discriminant $D(a, b, c) := b^2 - 4ac$.

   (a) If $D(a, b, c) > 0$, then the roots λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$. We then form the general solution

   $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

   (b) If $D(a, b, c) = 0$ then the roots of the characteristic equation are real and equal. Call this root λ. Then the two independent solutions are

   $$e^{\lambda t}, \;\text{ and }\; t\, e^{\lambda t},$$

   and we form the general solution:

   $$x(t) = c_1 e^{\lambda t} + c_2 t\, e^{\lambda t}.$$

(c) If $D(a, b, c) < 0$ then there are two complex conjugate roots of the characteristic equation λ and $\overline{\lambda}$. Then two *real* independent solutions can be formed from $\lambda = u + iv$ by using Euler's relation

$$e^{u+iv} = e^u[\cos(v) + i \sin(v)]$$

and taking the real and imaginary parts separately. Thus there are two *real* solutions $e^u \cos(v)$ and $e^u \sin(v)$, and the general solution has the form

$$x(t) = c_1 e^u \cos(v) + c_2 \sin(v).$$

3. In any of the three cases in (2) above, use the form of the general solution,

$$x(t) = c_1 x_1(t) + c_2 x_2(t),$$

together with the given initial data $x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0$, to find the constants c_1 and c_2 by solving the *algebraic* system

$$\begin{aligned} x_0 &= c_1 x^{(1)}(t_0) + c_2 x^{(2)}(t_0) \\ \dot{x}_0 &= c_1 \dot{x}^{(1)}(t_0) + c_2 \dot{x}^{(2)}(t_0). \end{aligned}$$

EXAMPLE 1: Solve the initial value problem

$$\ddot{x} + \dot{x} - 6x = 0, \quad x(0) = 5, \quad \dot{x}(0) = 0.$$

SOLUTION: The characteristic equation is $\lambda^2 + \lambda - 6 = 0$ which is easily factored:

$$(\lambda - 2)(\lambda + 3) = 0.$$

Hence the general solution has the form

$$x(t) = c_1 e^{-3t} + c_2 e^{2t}, \text{ with derivative } \dot{x}(t) = -3c_1 e^{-3t} + 2c_2 e^{2t}.$$

To find the solution which satisfies the initial conditions, we substitute the initial conditions given at $t = 0$ to yield a pair of simultaneous algebraic equations for the unknown constants c_1, and c_2.

$$\begin{aligned} x(0) &= c_1 e^0 + c_2 e^0 \\ \dot{x}(0) &= -3c_1 e^0 + 2c_2 e^0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} c_1 + c_2 &= 5 \\ -3c_1 + 2c_2 &= 0. \end{aligned}$$

Simple substitution, or the use of Cramer's Rule, leads to the solution:

$$c_1 = 2, \quad c_2 = 3,$$

and hence the solution of the initial value problem is

$$x(t) = 2e^{-3t} + 3e^{2t}.$$

EXAMPLE 2: Solve the initial value problem:

$$\ddot{x} + 6\dot{x} + 4x \;=\; 0, \;\; x(0) \;=\; 1, \;\; \dot{x}(0) \;=\; -3.$$

SOLUTION: The characteristic equation $\lambda^2 + 6\lambda + 4 = 0$ has solutions $\lambda_1 = -3-\sqrt{5}$ and $\lambda_2 = -3+\sqrt{5}$, which are obtained by using the quadratic formula. Hence the general solution of the homogeneous equation is

$$\begin{aligned} x(t) &= c_1 e^{(-3-\sqrt{5})t} + c_2 e^{(-3+\sqrt{5})t}, \;\; \text{with derivative} \\ \dot{x}(t) &= (-3-\sqrt{5})c_1 e^{(-3-\sqrt{5})t} + (-3+\sqrt{5})c_2 e^{(-3+\sqrt{5})t}. \end{aligned}$$

The equations for the unknown constants c_1 and c_2 are obtained by setting $t = 0, x(0) = 1$, and $\dot{x}(0) = -3$ to get

$$\begin{aligned} c_1 + c_2 &= 1 \\ (-3+\sqrt{5})c_1 + (-3-\sqrt{5})c_2 &= -3. \end{aligned}$$

This algebraic system has the solution $c_1 = c_2 = \frac{1}{2}$, and hence the solution of the initial value problem is

$$x(t) \;=\; \frac{1}{2}e^{(-3-\sqrt{5})t} + \frac{1}{2}e^{(-3+\sqrt{5})t} \;=\; e^{-3t}\left[\frac{e^{\sqrt{5}\,t} + e^{-\sqrt{5}\,t}}{2}\right]$$

or

$$x(t) \;=\; e^{-3t}\cosh(\sqrt{5}\,t).$$

EXAMPLE 3: Solve the initial value prolbem

$$\ddot{x} + 6\dot{x} + 9x \;=\; 0, \;\; x(1) \;=\; 0, \; \dot{x}(1) \;=\; 1$$

SOLUTION: The characteristic equation is $\lambda^2+6\lambda+9 \;=\; 0$ or $(\lambda+3)^2 \;=\; 0$. Hence the general solution has the form

$$\begin{aligned} x(t) &= c_1 e^{-3t} + c_2 t e^{-3t}, \;\; \text{with derivative} \\ \dot{x}(t) &= -3c_1 e^{-3t} + (1-3t)c_2 e^{-3t}. \end{aligned}$$

Substituting the initial conditions $x(1) \;=\; 0$ and $\dot{x}(1) = 1$ into these two equations gives us the appropriate system for the coefficients c_1 and c_2:

$$\begin{aligned} c_1 + c_2 &= 0 \\ -3c_1 - 2c_2 &= 1. \end{aligned}$$

This system is easily solved by using the first equation to derive $c_2 = -c_1$ and substituting in the second. The result is $c_1 = -1$ and $c_2 = 1$ and therefore the solution of the initial value problem is

$$x(t) = -e^3e^{-3t} + e^3e^{-3t} = -e^{-3(t-1)} + t\,e^{-3(t-1)}.$$

<u>EXAMPLE 4</u>: Solve the initial value problem

$$\ddot{x} - 4\dot{x} + 4x = 0, \;\; x(0) = 3, \; \dot{x}(0) = -1$$

<u>SOLUTION</u>: The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0$. Hence the general solution is

$$\begin{array}{lll} x(t) & c_1e^{2t} + c_2te^{2t}, \;\; \text{with derivative} & \\ \dot{x}(t) & = & 2c_1e^{2t} + (1+2t)c_2e^{2t}. \end{array}$$

Substituting the initial conditions into these two equations, we have

$$\begin{aligned} c_1 + 0c_2 &= 3 \\ 2c_1 + c_2 &= -1. \end{aligned}$$

This system has solution $c_1 = 3, \; c_2 = -7$ so that the solution we seek is

$$x(t) = 3e^{2t} - 7t\,e^{2t}.$$