EXAMPLES OF NON-HOMOGENEOUS
SECOND ORDER
LINEAR EQUATIONS

ANSWERS

In each of the following, we will NOT solve the homogeneous equation. We solve the non-homogeneous problem, give the general solution, and then solve for initial conditions if required.

1. \( \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = x \)

**Trial Solution:** \( y(x) = Ax + B \). \( y' = A \), \( y'' = 0 \). Substituting, \(-2A - 2B - 2Ax = x\) so, equating coefficients of like terms, \(-2A = 1 \Rightarrow A = -1/2\), and \(A + 2B = 0 \Rightarrow 2B = 1/2\) or \(B = 1/4\). Hence:

(a) Particular Solution: \( y(x) = (-1/2) x + (1/4) \).

(b) General Solution: \( y(x) = C_1 e^{-t} + C_2 e^{2t} - (x/2) + 1/4 \)

2. \( \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = 3x^2 \)

**Trial Solution:** \( y(x) = Ax^2 + Bx + C \). \( y' = 2Ax + B \), \( y'' = 2A \). Substituting, \(-2Ax^2 + (-2B - 2A)x + (2A - B - 2C) = 3x^2\) so, equating coefficients of like terms: \(-2A = 3\), \(-2A - 2B = 0\), and \(2A - B - 2C = 0 \Rightarrow A = (-3/2), B = (3/2)\) and \(C = (-9/4)\). Hence:

(a) Particular Solution: \( y(x) = (-3/2) x^2 + (3/2) x - (9/4) \).

(b) General Solution: \( y(x) = C_1 e^{-x} + C_2 e^{2x} - (3/2) x^2 + (3/2) x - (9/4) \)

3. \( y'' - 16y = \sin(2x) \),

with the initial conditions
\[
\begin{cases}
  y(1) = 1 \\
  y'(1) = 0 
\end{cases}
\]
given that the general solution of the homogeneous problem is \( y_h(x) = c_1 e^{-4x} + c_2 e^{4x} \).

**Trial Solution:** \( y(x) = A \cos(2x) + B \sin(2x) \), \( y' = -2A \sin(2x) + 2B \cos(2x) \), \( y'' = -4A \cos(2x) - 4B \sin(2x) \). Substituting, \(-4A \cos(2x) - 4B \sin(2x) - 16A \cos(2x) - 16B \sin(2x) = \sin(2x)\), so, equating coefficients of like terms: \( A = 0, -20B = 1 \), or \( B = (-1/20) \). Hence:
(a) Particular Solution: \( y(x) = (-1/20) \sin(2x) \).

(b) General Solution: \( y(x) = C_1 e^{-4x} + C_2 e^{4x} - (1/20) \sin(2x) \).

To solve the initial value problem, \( y'(x) = -4C_1 e^{-4x} + 4C_2 e^{4x} - (1/10) \cos(2x) \). Evaluating at \( x = 1 \):

\[
y(1) = C_1 e^{-4} + C_2 e^{4} - \left( \frac{1}{20} \right) \sin(2) = 1, \\
y'(1) = -4C_1 e^{-4} + 4C_2 e^{4} - \left( \frac{1}{10} \right) \cos(2) = 0.
\]

Solving these equations for \( C_1 \) and \( C_2 \) leads to the solution:

\[
y(x) = \left[ \left( \frac{1}{40} \right) \sin(2) e^{4} - \left( \frac{1}{80} \right) \cos(2) e^{4} \right] e^{-4x} + \left( \frac{1}{80} \right) \left( \cos(2) + 2 \sin(2) \right) e^{4x} - \left( \frac{1}{20} \right) \sin(2x).
\]

4. \( y'' + 16y = \sin(8x) \),

with the initial conditions

\[
\begin{align*}
y(1) &= 1 \\
y'(1) &= 0
\end{align*}
\]

**Trial Solution:** \( y(x) = A \cos(8x) + B \sin(8x) \), \( y' = -8A \sin(8x) + 8B \cos(8x) \), \( y'' = -64A \cos(8x) - 64B \sin(8x) \). Substituting, \( -80A \cos(8x) - 80B \sin(8x) = \sin(8x) \), so, equating coefficients of like terms: \( A = 0 \), \( -80B = 1 \), or \( B = (-1/80) \). Hence:

(a) Particular Solution: \( y(x) = (-1/80) \sin(8x) \).

(b) General Solution: \( y(x) = C_1 e^{-4x} + C_2 e^{4x} - (1/80) \sin(8x) \).

To solve the initial value problem, \( y'(x) = -4C_1 e^{-4x} + 4C_2 e^{4x} - (1/10) \cos(8x) \). Evaluating at \( x = 1 \):

\[
y(1) = C_1 e^{-4} + C_2 e^{4} - \left( \frac{1}{80} \right) \sin(8) = 1 \\
y'(1) = -4C_1 e^{-4} + 4C_2 e^{4} - \left( \frac{1}{10} \right) \cos(8) = 0.
\]

Solving these equations for \( C_1 \) and \( C_2 \) leads to the solution
5. \( y(x) = \left[ \left( \frac{1}{160} \right) \sin (8) - \left( \frac{1}{80} \right) \cos (8) + \left( \frac{1}{2} \right) \right] e^{-4x+4} + \left( \frac{1}{160} \right) \left[ (2 \cos (8) + \sin (8) + 80) \right] e^{4x-4} - \left( \frac{1}{80} \right) \sin (8x). \)

\[ z'' - 2z' + z = e^{-x}, \text{ with } z(0) = 0, \ z'(0) = 1, \]
given that the general solution to the homogeneous problem is \( z_h(x) = c_1 e^x + c_2 xe^x. \)

**Trial Solution:** \( z(x) = A e^{-x}, \) \( z' = -A e^{-x}, \) \( z'' = A e^{-x}. \) Substituting, \( A e^{-x} + 2A e^{-x} + A e^{-x} = e^{-x}, \) so, equating coefficients of like terms: \( 4A = 1, \) or \( A = (1/4). \) Hence:

(a) Particular Solution: \( z(x) = (1/4) e^{-x}. \)

(b) General Solution: \( z(x) = C_1 e^x + C_2 xe^x + (1/4) e^{-x}. \)

To solve the initial value problem, \( z'(x) = C_1 e^x + C_2 e^x + C_2 x e^x - (1/4) e^{-x}. \) Evaluating at \( x = 0: \)

\[ z(0) = C_1 e^0 + C_2 0 e^0 + \left( \frac{1}{4} \right) e^0 = 0 \text{ or } C_1 + \frac{1}{4} = 0 \]

\[ z'(0) = C_1 e^0 + C_2 e^0 + C_2 0 * e^0 - \left( \frac{1}{4} \right) e^0 = 1 \text{ or } C_1 + C_2 - \frac{1}{4} = 1. \]

Solving these equations for \( C_1 \) and \( C_2 \) yields \( C_1 = (-1/4), \) \( C_2 = (3/2), \) which leads to the solution \( z(x) = -(1/4) e^x + (3/2) x e^x + (1/4) e^{-x}. \)

6. \( \ddot{x} + 10 \dot{x} + 24 x = t + e^{2t}. \)

The easiest thing, from the point of view of bookkeeping, is to split the equation into two, one with right-hand side \( t \) and the other with right-hand side \( e^{2t}. \) On the other hand, we could treat it all at once and we do that here since we have seen other problems with simple right-hand sides before.

**Trial Solution:** \( x(t) = At + B + C e^{2t}, \) \( \dot{x}(t) = A + 2C e^{2t}, \) \( \ddot{x}(t) = 4C e^{2t}. \) Substituting, \( 4C e^{2t} + (10A + 20C e^{2t}) + 24At + 24B + 24C e^{2t} = t + e^{2t}, \) so, equating coefficients of like terms: \( 24A = 1, \) \( 10A + 24B = 0, \) and \( 48C = 1. \) Hence \( A = (1/24), \) \( B = (-5/288), \) \( C = (1/48). \)

(a) Particular Solution: \( x(t) = (1/24) t - (5/288) + (1/48) e^{2t}. \)
(b) General Solution: \( x(t) = C_1 e^{-4t} + C_2 e^{-6t}(1/24) t - (5/288) + (1/48) e^{2t} \).

7. \( \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 25 x = \cos (2t) \), with \( x(0) = 0 \), \( \ddot{x}(0) = \frac{1}{2} \).

given that the general solution of the homogeneous equation is \( x_h(t) = e^{-4t} [c_1 \cos (3t) + c_2 \sin (3t)] \).

What result do you get when you replace the right hand side with \( \cos (3t) \)?

**Trial Solution:** \( x(t) = A \cos (2x) + B \sin (2x) \) \( \dot{x}(t) = -2A \sin (2t) + 2B \cos (2t) \), \( \ddot{x}(t) = -4A \cos (2t) - 4B \sin (2t) \). Substituting, \(-4A \cos (2t) - 4B \sin (2t) - 16A \sin (2t) + 16B \cos (2t) + 25A \cos (2t) + 25B \sin (2t) = \cos (2t) \), so, equating coefficients of like terms: 

\[
21A + 16B = 1, \\
21B - 16A = 0,
\]

which has solutions \( A = (21/697) \), \( B = (16/697) \). Hence:

(a) Particular Solution: \( x(t) = (21/697) \cos (2t) + (16/697) \sin (2t) \).

(b) General Solution: \( x(t) = e^{-4t} [C_1 \cos (3t) + C_2 \sin (3t)] + (21/697) \cos (2t) + (16/697) \sin (2t) \).

To solve the initial value problem, since \( x(t) = e^{-4t} [C_1 \cos (3t) + C_2 \sin (3t)] + (21/697) \cos (2t) + (16/697) \sin (2t) \) and \( \dot{x}(t) = e^{-4t} [-4C_1 \cos (3t) - 4C_2 \sin (3t)] e^{-4t} [3C_2 \cos (2t) - 3C_1 \sin (3t)] + (32/697) \cos (2t) - (42/697) \sin (2t) \), substituting \( t = 0 \), we have \( C_1 + (21/697) = 0, 4C_1 + (633/1394)C_2 = -(1/2) \), which has the solution \( C_1 = -(21/697), C_2 = (155/1394) \) and so the solution of the initial value problem is

\[
x(t) = \left( \frac{155}{1394} \right) e^{-4t} \sin (3t) - \left( \frac{21}{697} \right) e^{-4t} \cos (3t) + \left( \frac{21}{697} \right) \cos (2t) + \left( \frac{16}{697} \right) \sin (2t).
\]

**REMARK:** Note that the solution can be written as the sum of a transient \( (155/1394) e^{-4t} \sin (3t) - (21/697) e^{-4t} \cos (3t) \) and a steady state \( (21/697) \cos (2t) + (16/697) \sin (2t) \).