EXAMPLES OF NON-HOMOGENEOUS SECOND ORDER LINEAR EQUATIONS

ANSWERS

In each of the following, we will NOT solve the homogeneous equation. We solve the non-homogeneous problem, give the general solution, and then solve for initial conditions if required.

1.
$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \;=\; x$$

Trial Solution: $y(x) = A\,x + B$. $y' = A$, $y'' = 0$. Substituting, $0 - A - 2\,B - 2\,A\,x = x$ so, equating coefficients of like terms, $-2\,A = 1 \Rightarrow A = -1/2$, and $A + 2\,B = 0 \Rightarrow 2\,B = 1/2$ or $B = 1/4$. Hence:

(a) Particular Solution: $y(x) = (-1/2)\,x + (1/4)$.

(b) General Solution: $y(x) = C_1 e^{(1-\sqrt{2})t} + C_2 e^{(1-\sqrt{2})t} - (x/2) + 1/4$

2.
$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \;=\; 3x^2$$

Trial Solution: $y(x) = A\,x^2 + B\,x + C$. $y' = 2\,A\,x + B$, $y'' = 2\,A$. Substituting, $-2\,A\,x^2 + (-2\,B - 2\,A)\,x + (2\,A - B - 2\,C) = 3\,x^2$ so, equating coefficients of like terms: $-2\,A = 3, -2\,A - 2\,B = 0$, and $2\,A - B - 2\,C = 0 \Longrightarrow A = (-3/2), B = (3/2)$ and $C = (-9/4)$. Hence:

(a) Particular Solution: $y(x) = (-3/2)\,x^2 + (3/2)\,x - (9/4)$.

(b) General Solution: $y(x) = C_1 e^{-x} + C_2 e^{2x} - (3/2)\,x^2 + (3/2)\,x - (9/4)$.

3.
$$y'' - 16\,y \;=\; \sin(2x)\,,$$

with the initial conditions

$$\begin{cases} y(1) = 1 \\ y'(1) = 0 \end{cases},$$

given that the general solution of the homogeneous problem is $y_h(x) = c_1 e^{-4x} + c_2 e^{4x}$.

Trial Solution: $y(x) = A\,\cos(2x) + B\,\sin(2x)$, $y' = -2\,A\,\sin(2x) + 2\,B\,\cos(2x)$, $y'' = -4\,A\,\cos(2x) - 4\,B\,\sin(2x)$. Substituting, $-4\,A\,\cos(2x) - 4\,B\,\sin(2x) - 16\,A\,\cos(2x) - 16\,B\,\sin(2x) = \sin(2x)$, so, equating coefficients of like terms: $A = 0, -20\,B = 1$, or $B = (-1/20)$. Hence:

(a) Particular Solution: $y(x) = (-1/20)\,\sin(2x)$.

(b) General Solution: $y(x) = C_1\,e^{-4x} + C_2\,e^{4x} - (1/20)\,\sin(2x)$.

To solve the initial value problem, $y'(x) = -4\,C_1\,e^{-4x} + 4\,C_2\,e^{4x} - (1/10)\,\cos(2x)$. Evaluating at $x = 1$:

$$y(1) = C_1\,e^{-4} + C_2\,e^{4} - \left(\frac{1}{20}\right)\sin(2) = 1\,,$$

$$y'(1) = -4\,C_1\,e^{-4} + 4\,C_2\,e^{4} - \left(\frac{1}{10}\right)\cos(2) = 0\,.$$

Solving these equations for C_1 and C_2 leads to the solution:

$$y(x) = \left[\left(\frac{1}{40}\right)\sin(2)\,e^{4} - \left(\frac{1}{80}\right)\cos(2)\,e^{4}\right]e^{-4x} + \left(\frac{1}{80}\right)\left[(\cos(2) + 2\,\sin(2))\,e^{-4}\right]e^{4x} - \left(\frac{1}{20}\right)\sin(2x).$$

4.

$$y'' + 16\,y = \sin(8x)\,,$$

with the initial conditions

$$\begin{cases} y(1) = 1 \\ y'(1) = 0 \end{cases}.$$

Trial Solution: $y(x) = A\,\cos(8x) + B\,\sin(8x)$, $y' = -8\,A\,\sin(8x) + 8\,B\,\cos(8x)$, $y'' = -16\,A\,\cos(8x) - 16\,B\,\sin(8x)$. Substituting, $-80\,A\,\cos(8x) - 80\,B\,\sin(8x) = \sin(8x)$, so, equating coefficients of like terms: $A = 0$, $-80\,B = 1$, or $B = (-1/80)$. Hence:

(a) Particular Solution: $y(x) = (-1/80)\,\sin(8x)$.

(b) General Solution: $y(x) = C_1\,e^{-4x} + C_2\,e^{4x} - (1/80)\,\sin(8x)$.

To solve the initial value problem, $y'(x) = -4\,C_1\,e^{-4x} + 4\,C_2\,e^{4x} - (1/10)\,\cos(8x)$. Evaluating at $x = 1$:

$$y(1) = C_1\,e^{-4} + C_2\,e^{4} - \left(\frac{1}{80}\right)\sin(8) = 1$$

$$y'(1) = -4\,C_1\,e^{-4} + 4\,C_2\,e^{4} - \left(\frac{1}{10}\right)\cos(8) = 0.$$

Solving these equations for C_1 and C_2 leads to the solution

$$y(x) = \left[\left(\frac{1}{160}\right)\sin(8) - \left(\frac{1}{80}\right)\cos(8) + \left(\frac{1}{2}\right)\right] e^{-4x+4} + \left(\frac{1}{160}\right)\left[(2\,\cos(8) + \sin(8) + 80)\right] e^{4x-4}$$
$$- \left(\frac{1}{80}\right)\sin(8x).$$

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5. $$z'' - 2\,z' + z = e^{-x}\,, \quad \text{with} \;\; z(0) = 0, \;\; z'(0) = 1\,,$$

given that the general solution to the homogeneous problem is $z_h(x) = c_1 e^x + c_2\, x e^x$.

Trial Solution: $z(x) = A\,e^{-x}, z' = -\,A\,e^{-x},\; z'' = A\,e^{-x}$. Substituting, $A\,e^{-x} + 2\,A\,e^{-x} + A\,e^{-x} = e^{-x}$, so, equating coefficients of like terms: $4\,A = 1$, or $A = (1/4)$. Hence:

(a) Particular Solution: $z(x) = (1/4)\,e^{-x}$.

(b) General Solution: $z(x) = C_1\,e^x + C_2\,x\,e^x + (1/4)\,e^{-x}$.

To solve the initial value problem, $z'(x) = C_1\,e^x + C_2\,e^x + C_2\,x\,e^x - (1/4)\,e^{-x}$. Evaluating at $x = 0$:

$$z(0) = C_1\,e^0 + C_2\,0\,e^0 + \left(\frac{1}{4}\right) e^0 = 0 \;\text{ or }\; C_1 + \frac{1}{4} = 0$$

$$z'(0) = C_1\,e^0 + C_2\,e^0 + C_2\,0 * e^0 - \left(\frac{1}{4}\right) e^0 = 1 \;\text{ or }\; C_1 + C_2 - \frac{1}{4} = 1.$$

Solving these equations for C_1 and C_2 yields $C_1 = (-1/4)$, $C_2 = (3/2)$, which leads to the solution $z(x) = -(1/4)\,e^x + (3/2)\,x\,e^x + (1/4)\,e^{-x}$.

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6. $$\ddot{x} + 10\,\dot{x} + 24\,x = t + e^{2t}\,.$$

The easiest thing, from the point of view of bookeeping, is to split the equation into two, one with right-hand side t and the other with right-hand side e^{2t}. On the other hand, we could treat it all at once and we do that here since we have seen other problems with simple right-hand sides before.

Trial Solution: $x(t) = A\,t + B + C\,e^{2t},\; \dot{x}(t) = A + 2\,C\,e^{2t}, \ddot{x}(t) = 4\,C\,e^{2t}$. Substituting, $4\,C\,e^{2t} + (10\,A + 20\,C\,e^{2t}) + 24\,A\,t + 24\,B + 24\,C\,e^{2t} = t + e^{2t}$, so, equating coefficients of like terms: $24\,A = 1$, $10A + 24\,B = 0$, and $48\,C = 1$. Hence $A = (1/24)$, $B = (-5/288), C = (1/48)$.

(a) Particular Solution: $x(t) = (1/24)\,t - (5/288) + (1/48)\,e^{2t}$.

(b) General Solution: $x(t) = C_1\, e^{-4t} + C_2\, e^{-6t}(1/24)\, t - (5/288) + (1/48)\, e^{2t}$.

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7. $$\frac{d^2x}{dt^2} + 8\,\frac{dx}{xt} + 25\,x \;=\; \cos(2t)\,, \quad \text{with} \;\; x(0) = 0, \quad \ddot{x}\,(0) = \frac{1}{2}\,.$$

given that the general solution of the homogeneous equation is $x_h(t) = e^{-4t}\,[c_1 \cos(3t) + c_2 \sin(3t)]$. What result do you get when you replace the right hand side with $\cos(3t)$?

Trial Solution: $x(t) = A\,\cos(2x) + B\,\sin(2x)$ $\dot{x}(t) = -2\,A\,\sin(2t) + 2\,B\,\cos(2t)$, $\ddot{x}(t) = -4\,A\,\cos(2t) - 4\,B\,\sin(2t)$. Substituting, $-4\,A\,\cos(2t) - 4\,B\,\sin(2t) - 16\,A\,\sin(2t) + 16\,B\,\cos(2t) + 25\,A\,\cos(2t) + 25\,B\,\sin(2t) = \cos(2t)$, so, equating coefficients of like terms: $21\,A + 16\,B = 1, 21\,B - 16\,A = 0$, which has solutions $A = (21/697), B = (16/697)$. Hence:

(a) Particular Solution: $x(t) = (21/697)\,\cos(2t) + (16/697)\,\sin(2t)$.

(b) General Solution: $x(t) = e^{-4t}\,[C_1\,\cos(3t) + C_2\,\sin(3t)] + (21/697)\,\cos(2t) + (16/697)\,\sin(2t)$.

To solve the initial value problem, since $x(t) = e^{-4t}\,[C_1\,\cos(3t) + C_2\,\sin(3t)] + (21/697)\,\cos(2t) + (16/697)\,\sin(2t)$ and $\dot{x}(t) = e^{-4t}\,[-4\,C_1\,\cos(3t) - 4\,C_2\,\sin(3t)]\,e^{-4t}\,[3\,C_2\,\cos(2t) - 3\,C_1\,\sin(3t)] + (32/697)\,\cos(2t) - (42/697)\,\sin(2t)$, substituting $t = 0$, we have $C_1 + (21/697) = 0$, $4\,C_1 + (633/1394)\,C_2 = -(1/2)$, which has the solution $C_1 = -(21/697), C_2 = (155/1394)$ and so the solution of the initial value problem is

$$x(t) \;=\; \left(\frac{155}{1394}\right)\, e^{-4t}\,\sin(3t) - \left(\frac{21}{697}\right)\, e^{-4t}\,\cos(3t) + \left(\frac{21}{697}\right)\,\cos(2t) + \left(\frac{16}{697}\right)\,\sin(2t).$$

REMARK: Note that the solution can be written as the sum of a transient $(155/1394)\,e^{-4t}\,\sin(3t) - (21/697)\,e^{-4t}\,\cos(3t)$ and a steady state $(21/697)\,\cos(2t) + (16/697)\,\sin(2t)$.