Solutions for Second Order Examples

(1)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y\,.$$

Characteristic equation: $\lambda^2 - \lambda - 2 = 0$

Characteristic roots: $\lambda_1 = -1,\ \lambda_2 = 2$

General Solution: $y(x) = c_1 e^{-t} + c_2 e^{2t}$

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(2)

$$y'' - 16\,y = 0\,,$$

with the two sets of initial conditions

$$(a)\quad \begin{cases} y(1) = 1 \\ y'(1) = 0 \end{cases}, \quad \text{and} \quad (b)\quad \begin{cases} y(1) = 0 \\ y'(1) = 1 \end{cases}.$$

For (a):

Characteristic equation: $\lambda^2 + 16 = 0$

Characteristic roots: $\lambda_1 = -4,\ \lambda_2 = 4$

General Solution: $y(x) = c_1 e^{-4x} + c_2 e^{4x}$

First Derivative: $y'(x) = -4c_1 e^{-4x} + 4c_2 e^{4x}$

Apply Initial Conditions: $y(1) = e^{-4}c_1 + e^4 c_2$

$y'(1) = -4e^{-4}c_1 + 4e^4 c_2$

Equations for Coefficients: $e^{-4}c_1 + e^4 c_2 = 1$

$-4e^{-4}c_1 + 4e^4 c_2 = 0$

Coefficients: $c_1 = \frac{1}{2}e^4,\ c_2 = \frac{1}{2}e^{-4}$

Solution of IVP: $y(x) = \frac{1}{2}e^4 e^{-4x} + \frac{1}{2}e^{-4}e^{4x}$

or $y(x) = \cosh(4x - 4)$.

For (b):

Characteristic equation:	$\lambda^2 + 16 = 0$
Characteristic roots:	$\lambda_1 = -4,\ \lambda_2 = 4$
General Solution:	$y(x) = c_1 e^{-4x} + c_2 e^{4x}$
First Derivative:	$y'(x) = -4c_1 e^{-4x} + 4c_2 e^{4x}$
Apply Initial Conditions:	$y(1) = e^{-4}c_1 + e^4 c_2$
	$y'(1) = -4e^{-4}c_1 + 4e^4 c_2$
Equations for Coefficients:	$e^{-4}c_1 + e^4 c_2 = 0$
	$-4e^{-4}c_1 + 4e^4 c_2 = 1$
Coefficients:	$c_1 = -\frac{1}{8}e^4,\ c_2 = \frac{1}{8}e^{-4}$
Solution of IVP:	$y(x) = -\frac{1}{8}e^4 e^{-4x} + \frac{1}{8}e^{-4}e4x$
	or $y(x) = \frac{1}{8}\sinh(4x-4)$.

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(3)

$$z'' - 2\,z' + z \;=\; 0\,, \quad \text{with} \;\; z(0) = 0, \quad z'(0) \;=\; 1\,.$$

Characteristic equation:	$\lambda^2 - 2\,\lambda + 1 = 0$
Characteristic roots:	$\lambda_1 = 1,\ \lambda_2 = 1$ (double root)
General Solution:	$z(x) = c_1 e^x + c_2 x\, e^x$
First Derivative:	$z'(x) = c_1 e^x + c_2 e^x + c_2 x e^x$
Apply Initial Conditions:	$z(0) = e^0 c_1$
	$z'(0) = e^0 c_1 + e^0 c_2$
Coefficients:	$c_1 = 0,\ c_2 = 1$
Solution of IVP:	$z(x) = x\, e^x$.

(4)

$$\ddot{x} + 10\,\dot{x} + 24\,x \;=\; 0\,.$$

Characteristic equation: $\lambda^2 + 10\,\lambda + 24 = 0$

Characteristic roots: $\lambda_1 = -6,\ \lambda_2 = -4$

General Solution: $y(x) = c_1 e^{-6t} + c_2 e^{-4t}$

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(5)

$$\frac{d^2x}{dt^2} - 6\,\frac{dx}{dt} + 25\,x \;=\; 0\,.$$

Characteristic equation: $\lambda^2 - 6\,\lambda + 25 = 0$

Characteristic roots: $\lambda_1 = \dfrac{6 + \sqrt{36 - 100}}{2} = \dfrac{6 + \sqrt{(-64)}}{2} = 3 + 4\,i,\ \lambda_2 = 3 - 4\,i$

General Solution: $y(x) = e^{3t}\ [c_1 \cos{(4t)} + c_2 \sin{(4t)}]$

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(6)

$$\frac{d^2x}{dt^2} + 8\,\frac{dx}{xt} + 25\,x \;=\; 0\,, \quad \text{with} \;\; x(0) = 0, \quad \left(\frac{dx}{dt}\right)(0) = \frac{1}{2}\,.$$

Characteristic equation: $\lambda^2 + 8\,\lambda + 25 = 0$

Characteristic roots: $\lambda_1 = \dfrac{-8 + \sqrt{64 - 100}}{2} = \dfrac{-8 + \sqrt{(-36)}}{2} = -4 + 3\,i,$

$\lambda_2 = -4 - 3\,i$

General Solution: $x(t) = e^{-4t}\ [c_1 \cos{(3t)} + c_2 \sin{(3t)}]$

First Derivative: $\dot{x}(t) = -4e^{-4t}\,[c_1 \cos{(3t)} + c_2 \sin{(3t)}]$

$\qquad +e^{4t}\,[-3c_1 \sin{(3t)} + 3c_2 \cos{(3t)}]$

Apply Initial Conditions: $x(0) = c_1, \dot{x}(0) = 3c_2$

Coefficients: $c_1 = 0,\ c_2 = \dfrac{1}{6}$

Solution of IVP: $x(t) = \dfrac{1}{6} e^{-4t}\ \sin{(3t)}$