M351 Sample Hour Examination–ANSWERS

1. (a) For
\[ \frac{dx}{dt} - \left( \frac{1}{t} \right) x = t^2, \quad t > 0, \]
the associated homogeneous equation is
\[ \frac{dx}{dt} - \left( \frac{1}{t} \right) x = 0. \]

(b) The general solution of the homogeneous equation is
\[ x(t) = C \exp \left( \int t \left( \frac{1}{s} ds \right) \right) = C e^{\ln(t)} = Ct. \]

(c) Since \( \frac{d}{dt} \left( \frac{t^3}{2} \right) = \frac{3}{2} t^2 \), substitution in the differential equation yields
\[ \frac{3}{2} t^2 - \frac{1}{t} \frac{t^3}{2} = \left( \frac{3}{2} - \frac{1}{2} \right) t^2 = t^2. \]
Hence the general solution is given by \( x(t) = Ct + \frac{t^3}{2}. \)

2. The separable equation
\[ \frac{dy}{dx} = y^2 - y - 2, \]
has constant solution obtained by setting the right hand side equal to zero and solving the quadratic equation \( y^2 - y - 2 = 0 \) which has roots \( y = -1 \) and \( y = 2 \). These give the constant solutions. Now separate variables to get
\[ \frac{1}{(y + 1)(y - 2)} \frac{dy}{dx} = 1, \]
or, using a partial fraction decomposition
\[ \frac{1}{3} \left[ \frac{1}{y - 2} - \frac{1}{y + 1} \right] \frac{dy}{dx} = 1. \]
Integrating, \( \frac{1}{3} \ln \left| \frac{y - 2}{y + 1} \right| = t + C \) or \( \left| \frac{y - 2}{y + 1} \right|^{1/3} = K e^x. \)
Letting \( K \) be either positive or negative and raising each side to the third power leads to the implicit solution
and imposing the initial condition \(y(0) = 0\) in this latter form leads to

\[
\frac{0 - 2}{0 + 1} = Re^{3^0} \text{ or } -2 = R,
\]

and thus to the final form \(\left(\frac{y - 2}{y + 1}\right) = -2e^{3x}\).
\[
\frac{dx}{dt} = -\frac{1}{N^2} \frac{dN}{dt} \text{ or } \frac{dN}{dt} = -N^2 \frac{dx}{dt}.
\]

Substituting into the Logistic Equation yields

\[
-N^2 \frac{dx}{dt} = rN - \frac{r}{K} N^2 \text{ or dividing by } N^2 \frac{dx}{dt} = -r x + \frac{r}{K}.
\]

This equation for \( x \) is linear with solution \( x(t) = C \exp (-rt) + \frac{1}{K} \). Note that the second term is a constant solution of the equation for \( x \).

(b) Since \( x = 1/N \) we have

\[
N(t) = \frac{K}{KCe^{-rt} + 1}
\]

and clearly, regardless of the value of \( C \neq 0 \), \( N(t) \rightarrow 0 \) as \( t \rightarrow \infty \).