PHASE PORTRAITS FOR FIRST ORDER TWO-DIMENSIONAL SYSTEMS

The figures given below show typical examples of the six possible phase portraits for constant coefficient two-dimensional linear systems. The geometric character of the phase portrait is determined by the nature of the eigenvalues of the system. In lecture, we have broken down the possibilities as follows:

1. λ_1 and λ_2 are real and unequal.
   (a) λ_1 and λ_2 have the same sign.
   (b) λ_1 and λ_2 have different signs.
2. There is a double root.
   (a) $A - \lambda I$ has rank 0.
   (b) $A - \lambda I$ has rank 1.
3. λ_1 and λ_2 are complex conjugate.
   (a) The real part of λ is 0.
   (b) λ_1 and λ_2 are purely imaginary.

The figures are listed in this order and are labeled. Note that the lines determined by the eigenvectors are pictured as heavy straight lines. Note that the *direction* of motion along the phase curves is determined in the case of real eigenvalues by their signs. So, for example, since in Case 1 both eigenvalues are negative, the motion is *toward* the origin and the proper node is *stable*. Similarly, in Case 2b since the eigenvalue is positive, the motion is *away* from the origin and the diagram is called a *source*, while were the eigenvalue negative, it would be called a *sink*.

Case 1a:

$$A = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$$

with eigenvalues $\lambda_1 = (-5+\sqrt{17})/2\,,\ \lambda_2 = (-5-\sqrt{17})/2\,$, and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ \\ (1/2)\,(1+\sqrt{17}) \end{pmatrix}\,,\ \mathbf{u} = \begin{pmatrix} 1 \\ \\ (1/2)\,(1-\sqrt{17}) \end{pmatrix}.$$

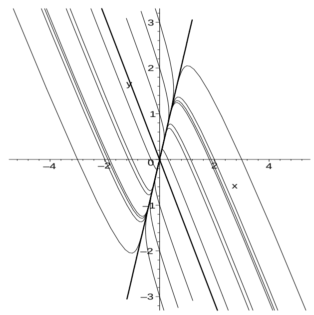

Figure 1: A Proper Node

Case 1b:

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -1\,,\ \lambda_2 = 3$, and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ \\ -2 \end{pmatrix}\,,\ \mathbf{u} = \begin{pmatrix} 1 \\ \\ 2 \end{pmatrix}.$$

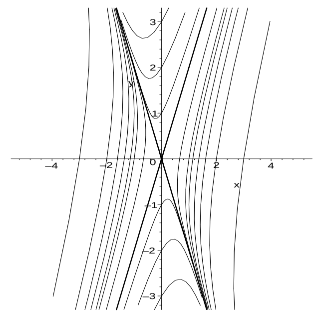

Figure 2: A Saddle

Case 2a:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

with double eigenvalue $\lambda = 2$ and corresponding eigenvector $\mathbf{v} = (-1, 1)^\top$,

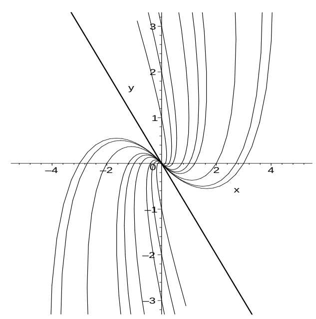

Figure 3: An Improper Node

Case 2b:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

with double eigenvalue $\lambda = 2$ and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}.$$

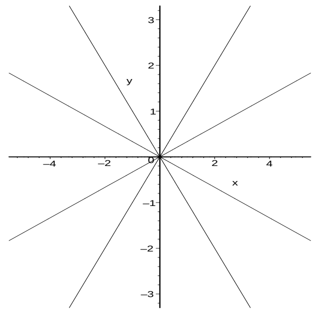

Figure 4: A Source

Case 3a:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

with pure imaginary eigenvalues $\lambda_1 = i\sqrt{2}$, $\lambda_2 = -i\sqrt{2}$, and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ \\ i\sqrt{2} \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 1 \\ \\ -i\sqrt{2} \end{pmatrix}$$

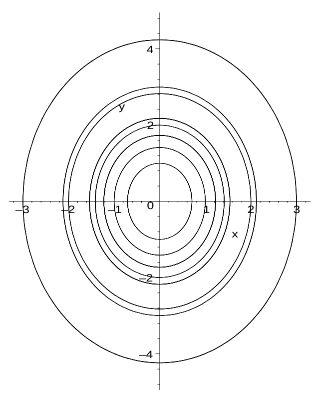

Figure 5: A Center

REMARK: Note that, from the differential equation, $\dot{x}_2 = -2\,x_1$ so that, if $x_1 > 0, x_2$ must be decreasing. Thus the motion along the elliptical paths is *clockwise*. Were the signs of the entries of A reversed, the direction along the paths would be as well and the motion would be *counterclockwise*. In the following case the real part of the eigenvalues are *positive* and the motion is *away* from the origin. The spiral is *unstable* (or more properly, the origin is unstable).

Case 3b:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

with pure imaginary eigenvalues $\lambda_1 = (1/2) + (i/2)\sqrt{7}\,, \lambda_2 = (1/2) - (i/2)\sqrt{7}$, and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} -(1/2) - (i/2)\sqrt{7} \\ 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -(1/2) + (i/2)\sqrt{7} \\ 1 \end{pmatrix}$$

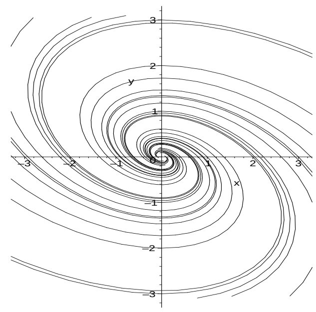

Figure 6: A Spiral

There is one degenerate case that should be noted that corresponds to the second order equation with the dependent variable missing. For example, consider the equation $\ddot{x} + 4\,\dot{x} = 0$. Writing this equation as a first order system we have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 4\,, \lambda_2 = 0\,$, and corresponding eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note that there are two distinct real eigenvalues but that this situation does not fit into the alternatives of Case 1.

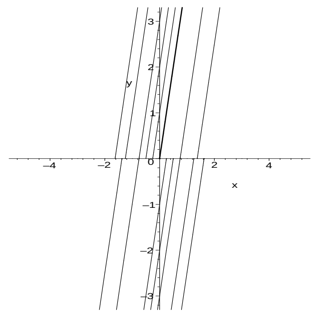

Figure 7: A Degenerate Case