Line Integrals and Vector Fields

The origin of the notion of line integral (really a path integral) comes from the physical notion of work. We will see that particular application presently. Our first task is to give a definition of what a path and line integrals are and see some examples of how to compute them.

1 Path Integrals

We start with a definition; later on we will try to justify the definition in terms of appropriate Riemann sums.

The problem that we want to treat is that of integrating a real-valued function of two or three variables along a curve in two- or three-dimensional space. We will give our definitions in terms of $\mathbb{R}^3$. We start with a given a function $f$ defined on a domain $D \subset \mathbb{R}^3$ so that $f : D \longrightarrow \mathbb{R}$, and a curve $c : r(t)$ with parameter domain an interval of the real line, $r : [a, b] \longrightarrow D$, $r(t) = (x(t), y(t), z(t))^\top$, $a \leq t \leq b$.

Since for all $t \in [a, b], r(t) \in D$, we can form the composition $f \circ r$ so that

$$(f \circ r)(t) = f(x(t), y(t), z(t)), \quad a \leq t \leq b.$$  

The path integral of the function $f$ along the curve $c$ is then defined as the integral of the scalar-valued function

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \frac{ds}{dt} \, dt = \int_a^b f(x(t), y(t), z(t)) \| \dot{r}(t) \| \, dt$$  

(1)

since $\frac{ds}{dt} = \| \dot{r}(t) \|$.

Notice that the factor $\| \dot{r}(t) \|$ is the speed of traversal of the curve as $t$ runs between endpoints of the interval $a$ and $b$.

To give a simple interpretation in a special case, recall that, if $f \equiv 1$, then the integral of a function $f$ over a domain $D \subset \mathbb{R}^2$,  

\[
\iint_D f(x,y) \, dx \, dy = \iint_D 1 \, dx \, dy,
\]
gives just the area of the domain \( D \). In the present case of the path integral, if \( f \equiv 1 \), the path integral is simply
\[
\int_a^b \| \mathbf{c}' \| \, dt.
\]
which is just the length of the curve! Let us begin by giving a couple of examples.

**Example 1.1** We start with an example in \( \mathbb{R}^2 \). Let \( f(x, y) = 2\sqrt{y} \) along the segment of the parabola \( y = x^2 \) lying between the points \((0,0)\) and \((1,1)\). Given the curve, we can parameterize it in any way that is convenient provided that the direction along the curve remains the same. This statement requires some proof and we will remark on that later. In the case of the parabola, a particularly easy parameterization is \((x(t), y(t)) = (t, t^2)\), \(0 \leq t \leq 1\). In this case \( \mathbf{r}(t) = (t, t^2) \) and \( \| \mathbf{r}'(t) \| = \sqrt{1 + 4t^2} \). So \( f(x(t), y(t)) = 2\sqrt{t^2} \). We can form and compute this path integral as indicated, using the substitution \( u = 1 + 4t^2 \):
\[
\int_c^d f(x,y) \, ds = \int_0^1 2t \sqrt{1 + 4t^2} \, dt = 2 \int_1^5 \frac{1}{8} \frac{1}{u^{\frac{1}{2}}} \, du
\]
\[
= \frac{1}{4} \left[ \frac{2}{3} \right]_1^5 \left[ u^{\frac{3}{2}} \right] = \frac{1}{6} (5\sqrt{5} - 1).
\]

**Example 1.2** We look at the integral of the function \( f(x, y, z) = x^2 + y^2 + z^2 \) along the helix \( \mathbf{r}(t) = (\cos(t), \sin(t), t)^\top \), \(0 \leq t \leq 2\pi\). Here, \( \dot{\mathbf{r}}(t) = (-\sin(t), \cos(t), 1)^\top \). The speed is then given by
\[
\| \dot{\mathbf{r}}(t) \| = \sqrt{(-\sin^2(t)) + (\cos^2(t) + 1} = \sqrt{2},
\]
while, \( f(x(t), y(t), z(t)) = \cos^2(t) + \sin^2(t) + t^2 = 1 + t^2 \), so that
\[
\int_c^d f(x,y,z) \, ds = \int_0^{2\pi} (1 + t^2) \sqrt{2} \, dt = \sqrt{2} \left[ t + \frac{t^3}{3} \right]_0^{2\pi}
\]
\[
= \sqrt{2} \left[ 2\pi + \frac{8}{3} \pi^3 \right]_0^{2\pi}
\]
2
Exercise 1.3 Let \( f(x,y) = 8\sqrt{y} + 2z \) and \( \mathbf{r}(t) := (1, t^2, e^{2t})^\top \). Compute the path integral of \( f \) along the curve.

In our discussion of curves in these lectures, we saw that the natural order of numbers on the real line induces a direction of motion along the curve, what we call an orientation of the curve. For example, if we parameterize the line between the point \( P \) and \( Q \) in \( \mathbb{R}^3 \) by \((1 - t)P + tQ, 0 \leq t \leq 1\) then a point moving along the line, starts at the point \( P \) when \( t = 0 \), and ends at the point \( Q \) when \( t = 1 \). On the other hand, if we move from \( t = 1 \) to \( t = 0 \), then the point moves from \( Q \) to \( P \). What happens to the path integral in such a case?

We consider the following example:

Example 1.4 Let \( f(x,y,z) = x + y + z \), and let \( \mathbf{c} \) be the straight line segment joining the points \( P = (0,0,0) \) and \( Q = (1,1,1) \). Then if \( \mathbf{r}(t) = (1 - t)(0,0,0)^\top + t(1,1,1)^\top, 0 \leq t \leq 1 \), we can compute

\[
\int_0^1 3t \, dt = \left. \frac{3}{2}t^2 \right|_0^1 = \frac{3}{2}(1^2 - 0^2) = \frac{3}{2},
\]

while if we move FROM \((1,1,1)^\top\) TO \((0,0,0)^\top\) the integral is

\[
\int_1^0 3t \, dt = \left. \frac{3}{2}t^2 \right|_1^0 = \frac{3}{2}(0^2 - 1^2) = -\frac{3}{2}.
\]

So in this example, moving in the opposite direction along the curve, that is, reversing the orientation, changes the sign of the integral. As another example, look at the Example (1.1), but now starting at \((1,1)\) to \((0,0)\).

Example 1.5 In Example (1.1), we claimed that we could use any convenient parameterization of the curve provided that the orientation is not changed. Let us look at three cases:

(a) Consider the curve parameterized by \((\cos (2t), \sin (2t))^\top\). For \( 0 \leq t \leq \frac{\pi}{4} \), the vector \((\cos (2t), \sin (2t))^\top\) traces out the portion of the unit circle in the first quadrant. As a simple example, we integrate the function \( f(x,y) = 2xy \) along this arc. Hence \( \mathbf{r}(t) = (\cos (2t), \sin (2t))^\top \) and so

\[
||\dot{\mathbf{r}}(t)|| = ||(-2\sin (2t), 2\cos (2t))^\top||
= \sqrt{4\sin^2 (2t) + 4\cos^2 (2t)} = 2.
\]
The path integral is then

\[
\int_{0}^{\frac{\pi}{4}} (2 \sin (2t) \cos (2t)) \, 2 \, dt = 2 \int_{0}^{\frac{\pi}{4}} \sin (4t) \, dt
\]

\[
= 2 \left[ -\frac{1}{4} \cos (4t) \right]_{0}^{\frac{\pi}{4}} = \frac{1}{2} [\cos (\pi) + 1]
\]

\[
= 1
\]

(b) The curve given by \( r(\tau) = (\cos (\tau^2), \sin (\tau^2))^\top \), \( 0 \leq \tau \leq \sqrt{\frac{\pi}{2}} \), describes the same path, that is, the first quarter of the unit circle. In this case, the relevant quantities are

\[
\|\dot{r}(\tau)\| = \|(−2\tau \sin (\tau^2), 2\tau \cos (\tau^2))^\top\|
\]

\[
= \sqrt{4\tau^2 \sin^2 (\tau^2) + 4\tau^2 \cos^2 (\tau^2)} = 2\tau.
\]

Then the path integral is

\[
\int_{0}^{\sqrt{\frac{\pi}{2}}} (2 \cos (\tau^2) \sin (\tau^2)) \, 2\tau \, d\tau = 2 \int_{0}^{\sqrt{\frac{\pi}{2}}} \sin (2\tau^2) \, \tau \, d\tau
\]

\[
= \left[ -\frac{1}{2} \cos (2\tau^2) \right]_{0}^{\sqrt{\frac{\pi}{2}}} = -\left[ \frac{1}{2} \cos (2 \frac{\pi}{2}) - \frac{1}{2} \right] = 1
\]

Exercise 1.6 In the example immediately above, start with the integral in part (a) and make the change of variable \( t = \frac{\tau^2}{2} \). Carry out the change of variable including the limits of integration and show that you get the same integral as in part (b) above.

(c) Looking at the same curve, suppose that we parameterize the quarter circle by \( r(t) = (\sin (t), \cos (t))^\top \). Note that in this case, as \( t \) runs from 0 to \( \frac{\pi}{2} \), the point tracing out the quarter circle runs between the points \((0, 1)\) and \((1, 0)\), that is, in the opposite direction from the direction of traversal in the preceding two parameterizations.

If \( c \) is given by this parameterization, then note that \( \dot{r}(t) = (\cos (t), -\sin (t))^\top \) and so
\[ \| \dot{r}(t) \| = 1, \quad \text{while} \quad \int_0^\pi (2 \sin(t) \cos(t)) \, dt = -\left[ \frac{1}{2} \cos(0) - \cos\left(\frac{\pi}{2}\right) \right] = -1. \]

Once again, reversal of the direction of the path changes the sign of the path integral.

We should now give a definition of the path integral by introducing the appropriate Riemann sums. All of the properties of the path integral follow easily from the definition in terms of these sums in a way completely analogous to the properties of the usual Riemann integral.

To motivate the definition of the path integral we consider sums \( S_N \) in the same way that is done in defining arc length. For simplicity, let \( r(t) \) be a continuously differentiable parameterization of a curve \( c \) on an interval \([a, b]\). This interval may be subdivided by introducing a partition \( a = t_0 < t_1 < t_2 < \ldots < t_N = b \). This leads to a decomposition of the curve into subcurves which we denote by \( c_i \), each defined on the appropriate subinterval \([t_i, t_{i+1}]\), \( i = 1, 2, \ldots, N-1 \). Denote the arc length of the subcurve \( c_i \) by \( \Delta s_i \). Then

\[ \Delta s_i = \int_{t_i}^{t_{i+1}} \| \dot{r}(t) \| \, dt. \]

When \( N \) is large, the arc length \( \Delta s_i \) is small and \( f(x, y, z) \) is approximately constant for points on \( c_i \). We consider the sums

\[ S_N = \sum_{i=0}^{N-1} f(x(t_i^*), y(t_i^*), z(t_i^*)) \Delta s_i, \]

where \( t_i^* \in [t_i, t_{i+1}] \). These sums are basically Riemann sums. We then make the following definition:

**Definition 1.7** Suppose that a function \( f : D \to \mathbb{R}^3 \) and a differentiable curve \( c \) with parameter domain \([a, b]\) whose graph lies in \( D \) are given and that \( f \circ c \) is continuous on \([a, b]\). If, as \( N \to \infty \) and \( \max(t_{i+1}, t_i) \to 0 \), \( \lim_{N \to \infty} S_N \) exists, then this limit is called the path integral and, as before, is written

\[ \lim_{N \to \infty} S_N = \int_a^b f(x(t), y(t), z(t)) \| \dot{r}(t) \| \, dt = \int_c f(x, y, z) \, ds. \]
It is often useful to introduce the notion of the **average value** of a function defined over a curve. If we look at the definition of the path integral in terms of Riemann sums, we know that the partial sum has the form

\[ S_N := \sum_{i=1}^{N} f(x(t_i^*), y(t_i^*), z(t_i^*))(s_i - s_{i-1}), \]

where we have \((s_i - s_{i-1}) = \int_{t_i}^{t_{i+1}} \|\dot{r}(t)\| \, dt.\) Then

\[ \sum_{i=1}^{N} (s_i - s_{i-1}) = L(c) \]

and so the quotient

\[ \frac{\sum_{i=1}^{N} f(x(t_i^*), y(t_i^*), z(t_i^*))(s_i - s_{i-1})}{\sum_{i=1}^{N} (s_i - s_{i-1})} \approx \frac{\sum_{i=1}^{N} f(x(t_i^*), y(t_i^*), z(t_i^*))(s_i - s_{i-1})}{L(c)}, \]

is the approximate average obtained by considering \(f(x(t_i^*), y(t_i^*), z(t_i^*))\) to be constant along each of the arcs \(\Delta s_i.\) The limit as \(N \to \infty\) of the sequence \(\{S_N\}_{N=1}^{\infty}\) is the average value of \(f\) along the curve \(c.\) We then can state

**Average Value of a Function on a Curve**

\[
\frac{\int_{a}^{b} f(x(t), y(t), z(t)) \|\dot{r}(t)\| \, dt}{\int_{a}^{b} \|\dot{r}(t)\| \, dt}
\]

**Example 1.8** If we consider the example (1.2), then we might think of the helix as a wire which is heated and interpret the function \(f(x, y, z) = x^2 + y^2 + z^2\) as giving the temperature at each point of the helical wire. Then the average temperature on the wire is given by
Exercise 1.9 Define the path $c$ in the $xy$-plane by $\mathbf{r}(t) = (30 \cos^3(t), 30 \sin^3(t))^\top$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Consider the function $f(x, y) = 1 + \frac{y}{3}$. Find the average value of the function over the given curve.

2 Vector Fields

The next integral we study is the line integral which is an integral which involves vector fields. Before we consider these integrals, we look at the idea of vector fields which we contrast with scalar fields. An example of the former is a region $D \subset \mathbb{R}^2$ at each point of which the acceleration of a particle moving in $D$ is given. In the latter case, a scalar field is illustrated by giving the temperature (a scalar!) at each point of $D$.

Vector and scalar fields are of great importance in applications in thermodynamics, fluid dynamics, acoustics, and electromagnetics among other applied areas. For us, vector fields are not completely new. We start with a definition, and then turn to some familiar examples.

Definition 2.1 Let $D \subset \mathbb{R}^n$. A vector field is a mapping $\mathbf{F} : D \rightarrow \mathbb{R}^n$.

In this definition, the domain of the vector field is considered a subset of $\mathbb{R}^n$ while the range of $\mathbf{F}$ is considered as a subset of an $n$-dimensional vector space which we take to be $\mathbb{R}^n$. Usually, this distinction is skipped over. In contrast to a vector field, a scalar field is a function $f : D \rightarrow \mathbb{R}$. These definitions immediately give us an entire class of vector fields, the so-called gradient fields.
Example 2.2 Suppose that we are given a scalar field \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = xe^{xy} + 3y^2 \). Then the gradient of \( f \) defines a vector field on all of \( \mathbb{R}^2 \). Specifically, the vector field is given by

\[
\nabla f = \left( e^{xy} + xy e^{xy}, xy e^{xy} + 6y \right)
\]

As another simple example consider:

Example 2.3 Let \( D \) be the unit disk in \( \mathbb{R}^2 \) and \( F : D - \{0\} \to \mathbb{R}^2 \) be given by

\[
F(x, y) = \left( \frac{y}{\sqrt{x^2 + y^2}} \hat{i} - \frac{x}{\sqrt{x^2 + y^2}} \hat{j} = \left( \frac{y}{\sqrt{x^2 + y^2}}, -\frac{x}{\sqrt{x^2 + y^2}} \right) \right).
\]

Remark: Note that the norm of each element in the range of \( F \), is

\[
\|F(x,y)\| = \sqrt{\left( \frac{y}{\sqrt{x^2 + y^2}} \right)^2 + \left( -\frac{x}{\sqrt{x^2 + y^2}} \right)^2}
\]

We can compute the corresponding vectors at each point in \( D \), so, for example, along the \( y \)-axis, the vector field is a vector of increasing magnitude as we approach the origin. For the positive \( y \)-axis, \( 0 \leq y \leq 1 \), \( F(0, y) = \left( \frac{y}{\sqrt{y^2}} \right)^\top = (1, 0)^\top \), while along the negative \( y \)-axis, \( F(0, y) = \left( \frac{y}{\sqrt{y^2}} \right)^\top = (-1, 0)^\top \) (remember that \( \sqrt{y^2} = |y| \) so that \( \frac{y}{|y|} = \text{sign}(y)! \)).

Now looking along the \( x \)-axis, for \( x > 0 \), \( F(x, 0) = \left( 0, -\frac{x}{\sqrt{x^2}} \right)^\top = (0, -1)^\top \) while for \( x < 0 \), \( F(x, 0) = (0, 1)^\top \).

If we look at the line \( x = y \), then

\[
F(x, y) = \left( \frac{y}{\sqrt{2y^2}}, -\frac{x}{\sqrt{2x^2}} \right) = \left( \frac{y}{\sqrt{y^2}}, -\frac{x}{\sqrt{x^2}} \right) = \left( \frac{\text{sign}(y)}{\sqrt{2}}, -\frac{\text{sign}(x)}{\sqrt{2}} \right)
\]

Exercise 2.4 Sketch this vector field on the unit disk.

\* \( D - \{0\} \) is the unit disk without the center point \( (0,0) \).
Example 2.5 A more familiar example is the vector field defined by Newton’s Law of Gravitation. Recall that the magnitude of the gravitational force between two masses, taking the center of mass of the larger body as the origin of coordinates, is given by \( \| \mathbf{F}(x, y, z) \| = \frac{mMG}{r^2} \). The direction of the force is in the direction from the point \((x, y, z)\) toward the center of the larger body of mass \(M\) i.e. in the direction \(-\frac{\mathbf{r}}{||\mathbf{r}||}\) or, writing \(r = ||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2}, -\frac{\mathbf{r}}{||\mathbf{r}||} = -\frac{\mathbf{r}}{r}\). Hence, the gravitation vector field is given by

\[
\mathbf{F}(x, y, z) = -\frac{mMG}{r^3} \mathbf{r}.
\]

One important relation that should be pointed out. If we define a scalar field by \(V(x, y, z) = \frac{mMG}{r}\) then we can look at the gradient of \(V\). If we differentiate \(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\) with respect to the variable \(x\), we obtain,

\[
\frac{\partial}{\partial x} \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = -\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = -\frac{x}{r^3}.
\]

Considering the symmetry of the function \(\frac{1}{r}\), we have

\[
\nabla V = (mMG) \begin{pmatrix}
\frac{\partial V}{\partial x} \\
\frac{\partial V}{\partial y} \\
\frac{\partial V}{\partial z}
\end{pmatrix} = -(mMG) \begin{pmatrix}
\frac{x}{r^3} \\
\frac{y}{r^3} \\
\frac{z}{r^3}
\end{pmatrix} = -(mMG) \frac{(x, y, z)}{r^3} = -(mMG) \frac{r}{r^3} \mathbf{r}.
\]

If we look at this result, we see that the vector field of the gravitational force is the gradient of a vector field. The scalar field \(V\) is called the potential of the field \(\mathbf{F}\). Whenever a vector field can be expressed as the gradient of a scalar field, we call that vector field conservative. This use of terminology will be justified in the next example. From this example, we see that it is important to understand just when a vector field is conservative and, moreover, how to find the scalar potential field.
Example 2.6 The total energy \( E \) of a particle of mass \( m \), moving in a conservative force field \( \mathbf{F} \) is defined to be the total of the kinetic and potential energies of the particle. Specifically, if \( \mathbf{F}(\mathbf{r}(t)) = -\nabla V(\mathbf{r}(t)) \), we have

\[
E(t) = \frac{1}{2} m \| \mathbf{r}'(t) \|^2 + V(\mathbf{r}(t)).
\]

We have called such a force field \textbf{conservative}. From this expression for the energy, it follows that

\[
\frac{d}{dt} E(t) = \frac{1}{2} m \frac{d}{dt} [\mathbf{r}''(t) \cdot \mathbf{r}'(t)] + \frac{d}{dt} V(\mathbf{r}(t)) = \frac{1}{2} m 2 [\mathbf{r}''(t) \cdot \mathbf{r}'(t)] + \nabla V(\mathbf{r}(t)) \cdot \mathbf{r}'(t)
\]

\[
= (m \mathbf{r}''(t) \cdot \mathbf{r}'(t)) - \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) - \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.
\]

Since the derivative \( \frac{dE}{dt} \) vanishes, the energy is constant. This is exactly what is meant by \textbf{conservation of energy}.

Now line integrals are integrals of the restriction of a vector field, \( \mathbf{f} \), to an oriented path. To define them, we go back to the notion of Riemann sum. Begin by dividing the curve \( \mathbf{c} \) into pieces. Each point of the subdivision is represented by a vector \( \mathbf{r}_i, i = 0, 1, \ldots, n \), and we write \( \Delta \mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i \). Now we form the \textit{dot product} \( \mathbf{f}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i \) which compares \( \Delta \mathbf{r}_i \) with the value of the vector field at \( \mathbf{r}_i \). Then we sum over all \( i \) to get a Riemann sum

\[
\sum_{i=1}^{n} \mathbf{f}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i.
\]

Taking limits as \( n \to \infty \) and \( \| \Delta \mathbf{r}_i \| \to 0 \) we have, provided the limit exists,

\[
\int_{\mathbf{c}} \mathbf{f} \cdot d\mathbf{r} := \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{f}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i.
\]

Example 2.7 Let \( \mathbf{f} = (1, 2)^T \) and let \( \mathbf{c} \) be the path consisting of the line segment \( \ell_1 \) joining (1,1) to (10,1) and \( \ell_2 \) that joining (10,1) to (10,10). To compute the integral, look at the line segments separately. On \( \ell_1, \Delta \mathbf{r}_i = (\Delta x, 0)^T \) and \( \mathbf{f} \cdot \Delta \mathbf{r}_i = (1, 2)^T \cdot (\Delta x, 0)^T = \Delta x \). Hence

\[
\int_{\ell_1} \mathbf{f} \cdot d\mathbf{r} = \int_{1}^{10} dx = 9.
\]

On \( \ell_2, \Delta \mathbf{r}_i = (0, \Delta y)^T \) and \( \mathbf{f} \cdot \Delta \mathbf{r}_i = (1, 2)^T \cdot (0, \Delta y)^T = 2\Delta y \). Hence

\[
\int_{\ell_2} \mathbf{f} \cdot d\mathbf{r} = \int_{1}^{10} 2 \Delta y = 18.
\]
\[ \int_{c_2} f \cdot dr = \int_2^{10} 2 \, dy = 18. \]

Hence \( \int_c f \cdot dr = 9 + 18 = 27. \)

A more complicated example is the following.

**Example 2.8** Let \( f \) be as in the previous example, but take for \( c \) the parabolic path \( y = (1/9) (x^2 - 2x + 10) \). We think of the curve as parameterized by \( x \) so that \( c : (t, t^2/9 - 2t/9 + 10/9)^\top \), \( 1 \leq t \leq 10 \). This is a curve with *the same endpoints* at the curve in the preceding example. Now, \( \dot{r}(t) = (1, (2t - 2)/9)^\top \) or, in differential form, \( dr = (1, (2t - 2)/9)^\top dt \). So

\[
\int_c f \cdot \dot{r} = \int_c (1, 2)^\top \cdot (1, (2t - 2)/9)^\top dt
= \int_1^{10} \left( \frac{4t}{9} + \frac{5}{9} \right) dt
= \left( \frac{200}{9} + \frac{50}{9} \right) - \frac{7}{9}
= \frac{243}{9}.
\]

This example shows that the value of the integral may well depend on the path.

The line integral of a vector field along a path has a physical interpretation in the case that \( f \) is a field due to a force and the curve is the path of a particle. Then the integral has the interpretation of work done by the force along the path \( c \).

**Example 2.9** Consider a mass on a flat table attached to a spring with spring constant \( k \) and whose other end is attached to a wall. We consider the frictionless case. Suppose that the spring is stretched 20 cm beyond the equilibrium position. We want to compute the work done by the spring. The force exerted by the spring is \( f = -k(x, 0)^\top \) where \( x \) is the displacement from equilibrium. In this experiment, the path is just the line from \( x = 20 \) to \( x = 0 \) and \( \Delta x \) is negative. Hence \( f \cdot \Delta r = -kx \Delta x \) so that

\[
\int_c f \cdot dr = \int_0^{20} (-kx) \, dx = \left. \frac{-kx^2}{2} \right|_0^{20} = \frac{k(20)^2}{2} = 200k.
\]

Another physical example is given by
Example 2.10 Compute the work done by gravity as a mass moves 8000 km along a circular arc at a height 10,000 km above the surface of the earth. In this case the arc of the circle is everywhere perpendicular to the radius vector. Since the gravitational force acts along the radius vector from the center of the earth, this vector field is always perpendicular to the arc. Hence $f(r) \cdot \Delta r = 0$ along the path. Hence the work done is zero. This is why satellites can remain in orbit without expending fuel once they are in orbit.

In the Example (2.8), we somewhat cavalierly used the parameterization of a curve to compute the line integral. How can we justify the computation. Suppose that we have a curve parameterized by $r(t) = (x(t), y(t), z(t))\top$, $a \leq t \leq b$. We start with the velocity vector $\dot{r}(t)$ and partition the parameter interval by points $\{t_i\}_{i=0}^n$. Then, on the interval $[t_i, t_{i+1}]$, $\Delta r_i = (\Delta x_i, \Delta y_i, \Delta z_i)\top$, where, for example $\Delta y_i := y(t_{i+1}) - y(t_i)$. Assuming the continuity of the vector function $t \mapsto \dot{r}(t)$, we can use the Mean Value Theorem to write that $\Delta r_i = r(t_{i+1}) - r(t_i) = \dot{r}(\xi_i) \Delta t_i$ where $t_i < \xi < t_{i+1}$. We form the Riemann sum

$$\sum_{i=1}^n f(r(\xi_i)) \cdot \Delta r_i = \sum_{i=1}^n f(r(\xi_i)) \cdot \dot{r}(\xi_i) \Delta t_i,$$

which, in the limit as $\|\Delta t_i\| \to 0$, yields the integral $\int_a^b f(r(t)) \cdot \dot{r}(t) \, dt$.

Example 2.11 Let $f(r) = (x+y, y)\top$ and take $c$ to be the quarter circle in the first quadrant with positive orientation. Use the parameterization $r(t) = (\cos(t), \sin(t))\top$, $0 \leq t \leq \pi/2$. Then

$$\int_c f \cdot dr = \int_0^{\pi/2} (\cos(t) + \sin(t), \sin(t))\top \cdot (-\sin(t), \cos(t))\top \, dt$$

$$= \int_0^{\pi/2} -\cos(t) \sin(t) - \sin^2(t) + \sin(t) \cos(t) \, dt$$

$$= -\int_0^{\pi/2} \sin^2(t) \, dt = -\frac{\pi}{4}.$$

Example 2.12 A particle is traveling along a circular helix given in parametric form by $r(t) = (\cos(t), \sin(t), 2t)\top$, $0 \leq t \leq 3\pi$, subject to the force field $f = (x, y, -xy)\top$. What is the work done by the force on the particle in moving it from the point $P : (1, 0, 0)$ to the point $Q : (1, 0, 6\pi)$?
In this case
\[
\int_c f \cdot dr = \int_0^{3\pi} f(r(t)) \cdot \dot{r}(t) \, dt
\]
\[
= \int_0^{3\pi} (\cos(t), 2t, -\cos(t) \sin(t))^\top \cdot (-\sin(t), \cos(t), 2)^\top \, dt
\]
\[
= \int_0^{3\pi} (-\cos(t) \sin(t) + 2t \cos(t) - 2 \cos(t) \sin(t)) \, dt
\]
\[
= \int_0^{3\pi} (-3 \cos(t) \sin(t) + 2t \cos(t)) \, dt = -4.
\]

REMARK: It is very common to see a different notation for a line integral, particularly in the plane \(\mathbb{R}^2\). Give functions \(P(x,y)\) and \(Q(x,y)\), defined in some region \(D \subset \mathbb{R}^2\), and an oriented curve \(c\), defined in the same region, we write
\[
\int_c f \cdot dr = \int_c (P(x,y) \, dx + Q(x,y) \, dy),
\]
where we have taken \(f = (P, Q)^\top\) and \(dr = (dx, dy)^\top\). As an example, we have

**Example 2.13** Evaluate the line integral \(\int_c xy \, dx - y^2 \, dy\) where \(c\) is the line segment from \((0,0)\) to \((2,6)\). To do this, parameterize the curve as \((1-t)(0,0)^\top + t(2,6)^\top = (2t,6t)^\top, 0 \leq t \leq 1\). Then
\[
\int_c xy \, dx - y^2 \, dy = \int_c (x, -y^2)^\top \cdot dr = \int_0^1 (12t^2, -36t^2)^\top \cdot (2,6)^\top \, dt
\]
\[
= \int_0^1 (24t^2 - 216t^2) \, dt = \int_0^1 -24t^2 \, dt = -64t^3\bigg|_0^1 = -64.
\]

Note that we could, as well, parameterized the line segment by \(r(t) = (t,3t), 0 \leq t \leq 2\). Then \(\dot{r}(t) = (1,3)^\top\) and
\[
\int_c xy \, dx - y^2 \, dy = \int_c (x, -y^2)^\top \cdot dr = \int_0^2 (3t^2, -9t^2)^\top \cdot (1,3)^\top \, dt
\]
\[
= \int_0^2 (-24t^2) \, dt = -8t^3\bigg|_0^2 = -64.
\]
This last computation agrees with the fact that the result is independent of the parameterization of the curve.

We now ask what happens when the vector field \( f \) has a potential. If we want to compute the line integral of \( f \cdot d\mathbf{r} \) along the curve \( c \) and if we assume that we have a parameterization (for example, we could parameterize the curve with respect to arc length), then the potential function \( g \), evaluated along the curve gives rise to a scalar function

\[
G(t) := g(x(t), y(t), z(t)), \quad a \leq t \leq b.
\]

First note that \( G(a) = g(P) \) and \( G(b) = g(Q) \) as a result of the parameterization of the curve. It is easy to compute the derivative of the function \( g \) by using the chain rule. Indeed, we have, writing as usual,

\[
\frac{dG}{dt} = \frac{\partial g}{\partial x}(r(t)) \frac{dx}{dt} + \frac{\partial g}{\partial y}(r(t)) \frac{dy}{dt} + \frac{\partial g}{\partial z}(r(t)) \frac{dz}{dt} = \text{grad } (g(r(t))) \cdot \dot{r}(t) = f(r(t)) \cdot \dot{r}(t).
\]

With this result we have

\[
\int_c f \cdot d\mathbf{r} = \int_a^b f(r(t)) \cdot \dot{r}(t) \, dt = \int_a^b \frac{dG}{dt} \, dt
\]

\[
= \left. G(t) \right|_a^b = G(b) - G(a) = g(Q) - g(P).
\]

or, in summary, we have an analog in this case of the Fundamental Theorem of Calculus

\[
\int_a^b \text{grad } (g(r(t))) \dot{r}(t) \, dt = g(Q) - g(P), \quad (2)
\]

It is important to understand that this last equation tells us that if \( f \) has a potential, or what is the same thing, if \( f \) is conservative, then the line integral \( \int_c f \cdot d\mathbf{r} \) is independent of the path of integration.

Example 2.14 Let \( P(x, y) = 2x + 3y \) and \( Q(x, y) = 3x - 2y \), and note that if \( g(x, y) = x^2 + 3xy - y^2 \) then \( g_x(x, y) = P(x, y) \) and \( g_y = Q(x, y) \). So if we want to compute the line integral over a curve \( c \) joining the points \((1, 3)\) and \((-2, 5)\) we do not have to produce a parameterization of the path. Indeed

\[
\int_c P \, dx + Q \, dy \, g(-2, 5) - g(1, 3) = -51 - 1 = -52.
\]
Example 2.15 A particle moves along the curve \( y = x^2 \) from the point \((1,1)\) to the point \((3,4)\). If the motion is caused by the force \( f = (x^2 + 2y, 2x + 2y)^\top \) applied to the particle, what is the work done by \( f \) on the particle?

The work is given by the line integral

\[
W = \int_c [(x^2 + 2y) \, dx + 2x \, dy].
\]

Since \( f = \text{grad} \left( \frac{x^3}{3} + 2xy + y^2 \right) \) we have

\[
W \left| _{(1,1)}^{(3,4)} \right. = 49 - \frac{10}{3} = \frac{137}{3}.
\]

REMARK: If we are given a closed curve with positive orientation, we can choose any two distinct point \( P \) and \( Q \) on the curve. Then the two portions of the close curve represent two paths, both joining \( P \) with \( Q \) and both having positive orientation. If the line integral of \( f \) is independent of the path, then the values of these two line integrals along these two paths are exactly the same. Let us call this common value \( V \). On the other hand, if we denote these two curves by \( c_1 \) and \( c_2 \) we traverse \( c \) by first traversing \( c_1 \) with positive orientation and then \( c_2 \) in the NEGATIVE orientation. Hence

\[
\oint_c f \cdot r = \int_{c_1} f \cdot r + \int_{c_2} f \cdot r = V - V = 0.
\]

Note that we have used the notation \( \oint \) to emphasize that the integral is being taken around a close curve.

As an example, consider

Example 2.16 Let \( f = (2x+4y, 4(x-y))^\top \) and let \( c \) be the closed curve consisting of the arc of the parabola \( y = x^2 \) joining \((0,0)\) with \((1,1)\), which we call \( c_1 \), and the line segment from \((1,1)\) to \((0,0)\). We introduce the parameterization \( r_1(t) \) for \( c_1 \) given by \( r_1(t) = (t, t^2)^\top, 0 \leq t \leq 1 \), so that \( \dot{r}_1(t) = (1, 2t)^\top \). For \( c_2 \) we take, as usual, \( r_2(t) = (1-t, 1-t)^\top, 0 \leq t \leq 1 \). Then \( \dot{r}_2(t) = (-1, -1)^\top \). Note that the curve \( c = c_1 + c_2 \).

Then, writing

\[
\oint_c f \cdot dr = \int_{c_1} P(x,y) \, dx + Q(x,y) \, dy + \int_{c_2} P(x,y) \, dx + Q(x,y) \, dy,
\]

we have
\[ \int_{c_1} P(x, y) \, dx + Q(x, y) \, dy = \int_0^1 [(2t + 4t^2) 1 + (4t - 4t^2) 2t] \, dt \]
\[ = \int_0^1 [2t + 12t^2 - 8t^3] \, dt = [t^2 + 4t^3 - 2t^4]_0^1 = 3, \]

and

\[ \int_{c_2} P(x, y) \, dx + Q(x, y) \, dy = \int_0^1 [2(1 - t) + 4(1 - t)] 1 + [4t - 4t] 1 \, dt \]
\[ = \int_0^1 6[(1 - t) dt = - \int_0^1 6(1 - t) \, dt = (6t - 3t^2)|_0^1 = -3. \]

Hence
\[ \oint f \cdot dr = 3 - 3 = 0. \]

We could anticipate this result by noting that \( \text{grad}(x^2 + 4xy - 2y^2) = (2x + 4y, 4x - 4y)^T \) so that the vector field is a gradient field.

**Remark:** Note that we could parameterize the straight line segment in the opposite direction by \( (t, t)^T, 0 \leq t \leq 1 \). In this case

\[ \int_{-c_1} P(x, y) \, dx + Q(x, y) \, dy = \int_0^1 6t \, dt = 3t^2|_0^1 = 3. \]

But since the orientation is opposite to the actual orientation of the closed path, this latter value should be *subtracted* from the first result.