UNDETERMINED COEFFICIENTS
for FIRST ORDER LINEAR EQUATIONS

This method is useful for solving non-homogeneous linear equations written in the form

$$\frac{dy}{dx} + ky = g(x),$$

where $k$ is a non-zero constant and $g$ is

1. a polynomial,
2. an exponential $e^{rx}$,
3. a product of an exponential and a polynomial,
4. a sum of trigonometric functions $\sin(\omega x), \cos(\omega x)$,
5. a sum of products $e^{rx} \sin(\omega x), e^{rx} \cos(\omega x)$,
6. a sum of terms $p(x), \sin(\omega x) + q(x) \cos(\omega x)$, where $p$ and $q$ are polynomials.

Here are a couple more examples.

**Example 1:**
Find the general solution of $y' - 4y = 8x^2$.
Here we take a trial solution to be a general polynomial of degree two

$$y_p(x) = Ax^2 + Bx + C.$$  

Then $y'_p(x) = 2Ax + B$ and substituting we have

$$(2Ax + B) - 4(Ax^2 + Bx + C) = 8x^2.$$  

Now, collecting like powers of $x$ we rewrite this equation as

$$-4Ax^2 + (2A - 4B)x + (B - 4C) = 8x^2,$$

and comparing coefficients of like terms on both sides of the equation gives $-4A = 8$, $2A - 4B = 0$, and $B - 4C = 0$, from which we see that $A = -2$, $B = -1$, and $C = -1/4$. Hence

$$y_p(x) = -2x^2 - x - \frac{1}{4},$$

and the general solution is
\[ y(x) = Ce^{4x} - 2x^2 - x - \frac{1}{4}. \]

**Example 2:**

Now consider the equation \( y' - 4y = 2x \cos(x) \). To find a particular solution, we take a trial solution of the form

\[
y_p(x) = (Ax + B) \cos(x) + (Cx + D) \sin(x),
\]

with

\[
y'_p(x) = Cx \cos(x) + (A + D) \cos(x) - Ax \sin(x) + (C + D) \sin(x).
\]

Then, again collecting like terms we have

\[
(X-4A) \cos(x) + (A+D-4B) \cos(x) - (A+4C) x \sin(x) + (C-B-4D) \sin(x) = 2x \cos(x).
\]

Equating coefficients leads to two sets of equations

\[
\begin{align*}
-4A + C &= 2, \\
A + 4C &= 0,
\end{align*}
\]

\[
\begin{align*}
A - 4B + D &= 0, \\
C - B - 4D &= 0.
\end{align*}
\]

In the first case, multiplying the second equation by 4 and adding the result to the first equation gives \( C = 2/17 \) and then the second equation gives \( A = -4C = -8/17 \). Substituting these value of \( A \) and \( C \) into the second set of equations yields

\[
\begin{align*}
-4B + D &= -\frac{8}{17}, \\
B + 4D &= \frac{2}{17},
\end{align*}
\]

which has solutions \( D = 0 \) and \( B = 2/17 \). Consequently the function

\[
y_p(x) = \left( -\frac{8}{17} x + \frac{2}{17} \right) \cos(x) + \frac{2}{17} x \sin(x),
\]

to which we add \( Ce^{4x} \) to get the general solution of the differential equation.