

UNDETERMINED COEFFICIENTS

for FIRST ORDER LINEAR EQUATIONS

This method is useful for solving non-homogeneous linear equations written in the form

$$\frac{dy}{dx} + k y = g(x),$$

where k is a non-zero constant and g is

1. a polynomial,
2. an exponential e^{rt} ,
3. a product of an exponential and a polynomial,
4. a sum of trigonometric functions $\sin(\omega t)$, $\cos(\omega t)$,
5. a sum of products $e^{rt} \sin(\omega t)$, $e^{rt} \cos(\omega t)$,
6. a sum of terms $p(t)$, $\sin(\omega t) + q(t) \cos(\omega t)$, where p and q are polynomials.

Here are a couple more examples.

Example 1:

Find the general solution of $y' - 4y = 8x^2$.

Here we take a trial solution to be a general polynomial of degree two

$$y_p(x) = Ax^2 + Bx + C.$$

Then $y'_p(x) = 2Ax + B$ and substituting we have

$$(2Ax + B) - 4(Ax^2 + Bx + C) = 8x^2.$$

Now, collecting like powers of x we rewrite this equation as

$$-4Ax^2 + (2A - 4B)x + (B - 4C) = 8x^2,$$

and comparing coefficients of like terms on both sides of the equation gives $-4A = 8$, $2A - 4B = 0$, and $B - 4C = 0$, from which we see that $A = -2$, $B = -1$, and $C = -1/4$. Hence

$$y_p(x) = -2x^2 - x - \frac{1}{4},$$

and the general solution is

$$y(x) = C e^{4x} + -2x^2 - x - \frac{1}{4}.$$

Example 2:

Now consider the equation $y' - 4y = 2x \cos(x)$. To find a particular solution, we take a trial solution of the form

$$y_p(x) = (Ax + B) \cos(x) + (Cx + D) \sin(x),$$

with

$$y'_p(x) = Cx \cos(x) + (A + D) \cos(x) - Ax \sin(x) + (C + D) \sin(x).$$

Then, again collecting like terms we have

$$(X-4A)x \cos(x) + (A+D-4B) \cos(x) - (A+4C)x \sin(x) + (C-B-4D) \sin(x) = 2x \cos(x).$$

Equating coefficients leads to two sets of equations

$$\begin{aligned} -4A + C &= 2, & A - 4B + D &= 0 \\ A + 4C &= 0, & C - B - 4D &= 0. \end{aligned}$$

In the first case, multiplying the second equation by 4 and adding the result to the first equation gives $C = 2/17$ and then the second equation gives $A = -4C = -8/17$. Substituting these value of A and C into the second set of equations yields

$$\begin{aligned} -4B + D &= -\frac{8}{17} \\ B + 4D &= \frac{2}{17}, \end{aligned}$$

which has solutions $D = 0$ and $B = 2/17$. Consequently the function

$$y_p(x) = \left(-\frac{8}{17}x + \frac{2}{17}\right) \cos(x) + \frac{2}{17}x \sin(x),$$

to which we add Ce^{4x} to get the general solution of the differential equation.