

Bernoulli and Riccati Equations

The following paper contains five exercises, the last of which is at the end of section 2. This is a set of problems you are to turn in. The due date is **October 15, 10:10 AM**.

1 Some Exercises

Solve the following Bernoulli equations using the substitution method described in lecture.

(1) $t^2 y' + 2ty - y^3 = 0 \quad t > 0.$

(2) $y' = \epsilon y - \sigma y^3, \quad \epsilon > 0, \sigma > 0.$ (Found in the study of stability of fluid flow.)

Find a second solution of each of the following Riccati equations given that the function φ in each case is a solution of the given equation.

(3) $\frac{dx}{dt} = x^2 - 1, \quad x(0) = 3$ with $\varphi(t) \equiv 1.$

(4) $\dot{x} = -x^2 + \frac{15}{4t^2}, \quad x(1) = 4$ with $\varphi(t) = \frac{a}{t}.$

2 The Quadratic Regulator

The Riccati equation occurs in the design of a feedback-control law for the linear-regulator problem with quadratic cost. A one-dimensional version of that problem is the following. We are given a system with output $x(t)$ and input (or control) $u(t)$ which are governed by the linear differential equation

$$\frac{dx}{dt} = a(t)x + b(t)u, \quad x(t_o) = x_o, \quad t_o \leq t \leq t_1.$$

The problem is to choose a control (or input function) $u = u(t)$ so as to minimize the cost or performance measure

$$C[u] = \frac{k}{2} x(t_1)^2 + \frac{1}{2} \int_{t_o}^{t_1} [w_1(t) x(t)^2 + w_2(t) u(t)^2] dt,$$

where $w_1(t)$ and $w_2(t)$ are given weighting functions and k is a non-negative constant.

Here is a specific example.

Example 2.1 *Water from a reservoir is being drawn off at an ever-increasing rate. In addition to compensating for this water loss, it is required to raise the level of the water in the reservoir above its initial height by pumping in fresh water. The cost of this operation per unit time is proportional to the square of the pumping rate, and we wish to adjust this rate so as to minimize the cost. The required level must be reached in a specified time. A set of model equations for this system, with all the parameters eliminated by appropriate scalings, can be stated as follows.*

Consider the state equation

$$\dot{x} = -v + u, \quad x(0) = 0, \quad x(t_1) = 1,$$

together with the cost function

$$C[u] = \int_0^{t_1} \frac{1}{2} u(t)^2 dt.$$

The height of the water above the initial level is measured by x , and we want to reach the height 1 in time t_1 . The pumping rate is given by $u(t)$ and $v(t)$ is the rate at which the water is drawn off. We could, for example, consider the case for which $v(t) = t$ and $t_1 = 1$.

The theory of the general problem shows that the optimal control $u(t)$, which minimizes $C[u]$, is realized through the *feedback control law*

$$u(t) = p^*(t) x(t), \quad p^*(t) = -\frac{b(t)}{w_1(t)} p(t),$$

where the function $p(t)$ satisfies the initial value problem for the Riccati equation:

$$\frac{dp}{dt} = -2a(t)p + \frac{b(t)^2}{w_2(t)} p^2 - w_1(t), \quad p(t_1) = k.$$

If one can find $p(t)$, the problem is completely solved, for we may then find the solution of

$$\frac{dx}{dt} = [a(t) + b(t)p^*(t)] x, \quad x(t_o) = x_o,$$

and subsequently evaluate

$$C[u] = \frac{k}{2} \int_{t_0}^{t_1} [w_1(t) + w_2(t) p^*(t)^2] x(t)^2 dt,$$

which will be a minimum.

Complete the following exercise:

(5) For the particular quadratic regulator problem

$$\begin{aligned} \frac{dx}{dt} &= s + u \\ C[u] &= \frac{3}{2}x(1)^2 + \frac{1}{2} \int_0^1 [3x(t)^2 + u(t)^2] dt, \end{aligned}$$

solve the problem completely, i.e., find the feedback control law, the optimal output $x(t)$ and the value of $C[u]$.