ANSWERS TO APPLIED PROBLEMS

(1) (a) Set up the coordinate system so that the initial position of the body is at the origin of coordinates with the positive x-axis pointing down. Since the only force acting on the body is the force due to gravitational attraction, the differential equation of the motion is, according to Newton's Second Law:

$$m\,\frac{dv}{dt} \;=\; m\,g.$$

Hence $v(t) = g\,t\,+\,C$ and, from the initial condition $v(0) = 0$ we conclude that the constant $C = 0$. Hence the velocity is given as a function of time as

$$v(t) = g\,t = 32\,t.$$

(b) Since $v(t) := \frac{dx}{dt}$ it follows that $x(t) = 16\,t^2 + C$ and the initial condition $x(0) = 0$ implies that $C = 0$. Hence the law of motion is

$$x(t) = 16\,t^2.$$

(c) To find the time, t_f, needed to reach the ground which is 100 feet from the starting point, simply solve the equation $100 = 16\;_f t^2$ for t_f. Thus

$$t_f = \sqrt{\frac{100}{16}} \approx 2.5\ \text{sec.}$$

(2) (a) Again, set up the coordinate system as in the previous problem. Since $w = m\;g$ the data of the problem yield that $m = 1/16$ slugs. The differential equation that we can derive from Newton's Second Law, taking into account the drag force as well as the force of gravity, is

$$\frac{dv}{dt}\;+\;2\,v\;=\;32.$$

Since the function $v(t) \equiv 16$ is a particular solution of the inhomogeneous equation and $v_h(t) = C\,e^{-2t}$ is the general solution of the homogeneous problem, the general solution of the inhomogeneous problem is

$$v(t) \;=\; C\, e^{-2t} \;+\; 16.$$

Using the initial condition $v(0) \;=\; 0$ yields $C \;=\; -16$. Consequently, the limiting velocity of the ball is:

$$\lim_{t\to\infty} v(t) \;=\; \lim_{t\to\infty} (-16\, e^{-2t} \;+\; 16) \;=\; 16 \;\text{ ft / sec.}.$$

(b) By simple integration we find that the position of the ball at any time t is given by

$$x(t) \;=\; 8e^{-2t} + 16t \;+\; C.$$

The initial condition $x(0) = 0$ yields a value of $C = -8$. To find the time of impact we need only solve the equation

$$3000 \;=\; 8e^{-2t_f} + 16t_f - 8.$$

The approximate solution is $t_f \approx 188$ sec.

(3) Since the body is thrown up from the surface, we choose a different coordinate system with the origin at ground level where the ball starts. We choose the upward direction as positive.

(a) To derive the equation of motion, we must take into account the two forces acting to impede the motion, that due to gravity, and that due to the drag. Newton's Second Law then yields the equation

$$m\,\frac{dv}{dt} \;=\; -m\,g \;-\; k\,v,$$

or

$$\frac{dv}{dt} \;+\; \frac{k}{m} \;=\; -g.$$

(b) The equation of part (a) is again a non–homogeneous linear equation with constant right-hand side. A particular constant solution is $-mg/k$ and so the general solution is

$$v(t) \;=\; C\, e^{-(k/m)t} - \frac{mg}{k}.$$

Using the initial condition yields $C = v_0 + (mg/k)$ so that the final expression is

$$v(t) = (v_0 + \frac{mg}{k})e^{-(k/m)t} - \frac{mg}{k}.$$

(c) At the time, t_M at which the body reaches the maximum height, $v(t_M) = 0$. Hence to find this time we must solve

$$v_0 + \frac{mg}{k}e^{-(k/m)t_M} - \frac{mg}{k} = 0.$$

This leads to

$$e^{-(k/m)t_M} = \frac{1}{1 + \frac{v_0 k}{mg}}$$

and solving for t_M we get $t_M = (m/k)\ \ln(1 + (v_0 k)/mg)$.

(4) From the description of the law of infection, the number of uninfected individuals is $(100 - I)$ and so the differential equation governing the epidemic is

$$\frac{dI}{dt} = k\, I\, (100 - I)$$

where k is the (positive) proportionality constant.

(a) This is a standard problem in separation of variables. The partial fraction decomposition is

$$\frac{1/100}{I} + \frac{1/100}{100 - I} = \frac{1}{(100 - I)\, I}$$

and integration yields $\ln|I| - \ln|100 - I| = 100\, k\, t$. Since both I and $(100 - I)$ are positive, we may drop the absolute value signs and the usual exponentiation leads to

$$\frac{I}{(100 - I)} = C\, e^{100kt}.$$

The initial conditions yield $C = 1/99$, and, solving the equation for I then leads to the expression

$$I = \frac{100e^{100kt}}{99 + e^{100kt}}.$$

(b) If the constant of proportionality k has value 0.01 when t is measured in days, then the expression for I is just

$$I = \frac{100e^t}{99 + e^t}.$$

The computation of $I'(t)$ at the end of each day for the first 9 days then requires the evaluation of 9 values of I and then substitution of these values into the original differential equation, clearly a job for MAPLE. You can write a little MAPLE program like the following. (Note: I is a protected symbol which is why J was chosen instead, r stands for "rate", and R(i) stands for the required rate I' at different times.)

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> J:=(100*exp(t))/(99+exp(t));
> r:=(0.01)*J*(100-J);
> for i from 1 to 9
    do
       R(i):=evalf(subs(t=i,r)):
    od;
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The output (rounded to the nearest integer) is 3,6,14,23,24,16,8,3,1. I wonder who the one lucky person was?

(5) (a) The differential equation

$$\frac{dx}{dt} = C\ x(t)\ \cos(t),$$

is separable and can be solved easily. The separated form is

$$\frac{1}{x}\ \frac{dx}{dt} = C\ \cos(t)$$

which yields the general solution $x(t) = K\ e^{(C\ \sin(t))}$. With the initial condition $x(0) = x_0$, we have $K = x_0$ in this expression.

(b) The usual necessary condition that a function has an extreme point is that the derivative vanish. So from the differential equation and the previous result we have

$$x_0\ e^{C\ \sin(t)}\ cos(t)\ =\ 0.$$

Since the exponential function never vanishes, the zeros coincide with the zeros of $\cos(t)$, namely at odd multiples of $\pi/2$. Clearly the distance between maxima is 2π, while the maximum value first occurs at $\pi/2$ and has value $x_0\, e^c$ while the minimum values are $x_0\, e^{-c}$.