Some Remarks on Taylor’s Theorem

Suppose that \( f \) is a real-valued function of a real variable and that it has derivatives of all orders up to \( n \) at a point \( a \). Then we can consider the Taylor polynomial of order \( n \) about \( a \), namely

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

and we can ask how well it approximates the function \( f \), or equivalently, how close the remainder

\[
R_n(x) = f(x) - T(x)
\]

is to zero. What is true is that as \( x \to a \), the remainder \( R_n(x) \) tends to zero faster than \((x-a)^n\). That is the content of the following:

**Theorem 0.1** If \( f^{(n)}(a) \) exists, then

\[
\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0.
\]

**Proof:** We prove the theorem by induction on \( n \). Indeed, se have

\[
\lim_{x \to a} \frac{R_1(x)}{(x-a)} = \lim_{x \to 0} \left[ \frac{f(x) - f(a)}{(x-a)} - f'(a) \right] = f'(a) - f'(a) = 0.
\]

Hence the theorem is true for \( n = 1 \).

Assume that the statement is true for \( n = k \). Then, assuming that \( f^{(n+1)}(a) \) exists, we set \( g = f' \). Now

\[
R_{k+1}(x) = f(x) - f(a) - f'(a) (x-a) - \frac{f''(a)}{2!} (x-a)^2
\]

\[
- \cdots - \frac{f^{(k+1)}(a)}{(k+1)!} (x-a)^{k+1},
\]

so that

\[
\frac{d}{dx} [R_{k+1}(x)] = f'(x) - f'(a) - f''(a) (x-a) - \cdots - \frac{f^{(k+1)}(a)}{k!} (x-a)^{k}
\]

\[
= g(x) - g(a) - g'(a) (x-a) - \cdots - \frac{g^{(k)}(a)}{k!} (x-a)^k.
\]
By the induction hypothesis,
\[
\lim_{x \to a} \frac{d}{dx} \left[ R_{k+1}(x) \right] (x-a)^k = 0.
\]

Since
\[
\lim_{x \to a} \frac{d}{dx} \left[ R_{k+1}(x) \right] = \lim_{x \to a} \frac{d}{dx} \left[ R_{k+1}(x) \right] = 0,
\]
it follows from l’Hôpital’s rule that
\[
\lim_{x \to a} R_{k+1}(x) (x-a)^k = 0,
\]
and so the theorem is true when \( n = k+1 \).

The fact that \( R_n(x) \to 0 \) as \( x \to a \) is often expressed by the notation that \( R_n(x) = o((x-a)^n) \), read \( R_n(x) \) is “little oh” of \((x-a)^n\). In general \( f = o(h^n) \) means \( f/h^n \to 0 \) as \( h \to 0 \).

Now we move to the multidimensional case.

**Theorem 0.2** Let \( C \subset \mathbb{R}^n \) be convex and suppose that \( F : C \to \mathbb{R} \). Consider the function
\[
\phi(t) := F(x_1 + t(x_2 - x_1)), \quad x_1, x_2 \in C.
\]
If \( \phi \) is \( n \)-times continuously differentiable at the right at \( t = 0 \) and \( \phi' \) is continuous on the interval \([0, \alpha)\) for some \( \alpha > 0 \), then
\[
F(x_1 + h) = F(x_1) + \sum_{k=1}^{n} \frac{1}{k!} F^{(k)}(x_a) h^k + r(x_1, h),
\]
where \( h = t(x_2 - x_1) \) and \( r(x_1, h) = o(\|h\|^n) \).

**Proof:** The preceding proof, repeated, *mutatis mutandis* for \( \phi \) yields
\[
\phi(t) = \phi(0) + t \phi'(0) + \cdots + \frac{t^n}{n!} \phi^{(n)}(0) + \rho(x_a, x_2, t),
\]
where \( \rho(x_a, x_2, t) = o(t^n) \) as \( t \to 0 \). Setting \( \rho(x_a, x_2, t) = r(x_1, h) \), we have
\[
0 = \lim_{t \to 0} r(x_1, h) = \|x_2 - x_1\|^n \lim_{h \to 0} \frac{r_1(x_1, h)}{\|h\|^n},
\]
that is \( \lim_{t \to 0} r(x_1, h) \to 0 \).

Furthermore, since
\[
\phi^{(ki)}(0) = F^{(k)}(x_1)(x_2 - x_1)^k, \quad k = 1, 2, \ldots, n,
\]

the series above for \( \varphi \) has the form

\[
F(x_1 + h) = F(x_1) + \sum_{k=1}^{n} F^{(k)}(x_1) h^k = o(\|h\|^n).
\]

Note that for \( n = 1 \) we can write

\[
F(y) = F(x) + \nabla F(x)(y - x) + o(\|y - x\|).
\]

In terms of the excess function we have

\[
E(x, y) = F(y) - F(x) + \nabla F(x) \cdot (y - x) = o(\|y - x\|),
\]

or

\[
\lim_{y \to x} \frac{E(x, y)}{\|y - x\|} = 0.
\]