Orderings on Sets

We are going to be particularly concerned with certain binary relations which are called orderings of which there are several types. Such relations are used in economics to describe preferences of various agents. Thus, for example, suppose that an \( n \)-vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) represents a “bundle” of goods available to consumers, \( x_i \) representing the amount of good \( i \) in the bundle. Thus, for example, if the first component represents the number of refrigerators measured in units while the second component represents wheat measured in bushels then \((2, 3.659, \ldots) \in \mathbb{R}^n\) is a bundle of goods consisting, among other things of two refrigerators and 3.659 bushels of wheat.

In describing consumer behavior, we generally make the assumption that one and only one of the following alternatives holds:

1. a bundle \( \mathbf{x} \) is preferred to the bundle \( \mathbf{y} \);
2. the consumer is indifferent in the choice of bundles;
3. the bundle \( \mathbf{y} \) is preferred to the bundle \( \mathbf{x} \).

This situation calls for some structure of preference.

**Definition 1.1** A binary relation \( \mathcal{R} \) in a set \( A \) is said to be a preorder on \( A \) if it is reflexive and transitive, i.e.,

(a) \( \forall a \in A, a \mathcal{R} a. \)

(b) If \( a \mathcal{R} b \) and \( b \mathcal{R} c \) then \( a \mathcal{R} c. \)

A set, together with a definite preorder is called a preordered set. It is traditional to write a preorder with the symbol \( \prec \). Thus “\( a \) precedes \( b \)”, or “\( b \) is preceeded by \( a \)”, or, in economics, “\( b \) is preferred to \( a \)” is written \( a \prec b \). The symbol \((A, \prec)\) denotes a preordered set. Notice that if \( B \subseteq A \) and if \( A \) is preordered by \( \prec \) then this preorder induces a preorder on \( B \).

**Example 1.2** (a) In any set, the relation \( \Delta \) is a preorder and \( a \prec b \) means \( a = b \).

Note that we no not assume that any two elements can be compared. In other words, we do not require that either \( a \prec b \) or \( b \prec a \) for all \( a, b \in A \).

(b) In the set \( \mathbb{R} \) the relation \( \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\} \) is a preorder. On the other hand \( \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x < y\} \) is not a preorder. (Why?)
(c) (IMPORTANT!) For any set $X$, consider the power set $\mathcal{P}(X)$. The relation $A \prec B$ defined by

$$A \prec B \iff A \subset B$$

is a preordering of $\mathcal{P}(X)$. In this particular case, we say that $\mathcal{P}(X)$ is preordered by inclusion.

There is a standard terminology pertaining to preordered sets.

**Definition 1.3** Let $(A, \prec)$ be preordered. Then

(a) $m \in A$ is called a **maximal element** in $A$ provided, for all $a \in A$ $m \prec a \Rightarrow a \prec m$.

In other words, if either no $a$ follows $m$ or if each $a$ that follows $m$ also precedes $m$.

(b) $a_o \in A$ is called an **upper bound** for a subset $B \subset A$ if $\forall b \in B b \prec a_o$.

(c) $B \subset A$ is called a **chain** in $A$ if each two elements in $B$ are related.

**Example 1.4** (a) In the preceding example part (a), each element is maximal. No subset of $A$ containing at least two elements has an upper bound. Thus a maximal element in $A$ need not be an upper bound for $A$.

(b) In part (b) of the preceding example, there is no maximal element. Every bounded set has many upper bounds.

By putting other conditions on a preordering, different types of orders can be obtained.

**Definition 1.5** If a preordering on $A$ satisfies the additional property of antisymmetry, i.e.,

$$a \prec b \text{ and } b \prec a \Rightarrow a = b,$$

then it is called a **partial ordering**. In this case $A$ is called a partially ordered set. A partially ordered set that is also a chain is called a **totally ordered set**.
Example 1.6 The set $\mathcal{P}(X)$ is partially ordered by inclusion since the preorder is also antisymmetric. In general, this set is not a chain. In this context, what does a chain look like? One example is the following: for each $n \in \mathbb{Z}$, let $A_n \subset \mathcal{P}(X)$ and suppose that, for each $n A_{n+1} \subset A_n$ and $A_n \neq A_{n+1}$. Then the set of subsets $\{A_n\}_{n=1}^{\infty}$ constitutes a chain in $\mathcal{P}(X)$ with respect to the partial ordering of inclusion.

Example 1.7 Let $X = [-1, 1] \subset \mathbb{R}$ and let $A_n = ]-\frac{1}{n}, \frac{1}{n}],$ $n = 1, 2, \ldots$. Then this set forms a chain, namely

$$[-1, 1] \supset [-\frac{1}{2}, \frac{1}{2}] \supset [-\frac{1}{3}, \frac{1}{3}] \supset \cdots$$

As another, and important, example we consider the following.

Example 1.8 Consider the set $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then we can introduce a partial order $\prec$ in $\mathbb{R}^n$ by

$$x \prec y \iff x_i \leq y_i \text{ for all } i = 1, 2, \ldots, n.$$ 

Here $\leq$ is the usual ordering on the real line. Note that this is certainly a reflexive, transitive, and anti-symmetric relation so that $\prec$ is indeed a partial ordering of $\mathbb{R}^n$.

Note further that not every two elements can be compared. Thus, for example, in $\mathbb{R}^2$, the vectors $(1, 2) and (2, 1)$ are not comparable.

This last example contrasts with the usual ordering $\leq$ on $\mathbb{R}$ where every element can be compared. This leads to an important special case of a partial order.

Definition 1.9 Let $A$ be a set. A total or linear order on the set $A$ is a partial order $\prec$ such that, $\forall x, y \in A, x \neq y$ either $x \prec y$ or $y \prec x$ whenever $x$ and $y$ are both in the domain and range of the order relation.

The usual order on the real line is the obvious example. We remark that, in this terminology, a chain in a partially ordered set is a totally ordered family.

Here is another which is important in a number of applications including integer programming algorithms.
Example 1.10 (Lexicographical Order) Let $X$ be the set of all infinite sequences of real numbers. Define a relation $\prec_L$ on $X$ by

$$a \prec_L b$$
provided, for the smallest integer $i_o \ni a_{i_o} \neq b_{i_o}$, $a_{i_o} < b_{i_o}$.

This order is called lexicographical order since it is the same kind of order used in common dictionaries. In face, this order is a total, or linear, order. Indeed, it is clearly reflexive. To check transitivity, note that if $a \prec_L b$ and $b \prec_L c$ then form minimal $i_o$ and $j_o$, we have $a_{i_o} < b_{i_o}$ and $b_{j_o} < c_{j_o}$. If $i_o < j_o$ then $a_{i_o} < b_{i_o} = c_{j_o}$ and so $a \prec_L c$. On the other hand, if $j_o < i_o$ then $a_{j_o} = b_{j_o} < c_{j_o}$ and so, again, $a \prec_L c$.

The property of anti-symmetry is easy to check. Finally, since any two sequences can be compared, the partial ordering is indeed a total order.