Some Examples and Properties of Polar Cones

Section 1: Introduction

Here is some supplemental material on polar cones. Recall, that if $C$ is a subset of $\mathbb{R}^n$ then its polar cone $C^\ast$ is the cone

$$C^\ast = \{ y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0 \text{ for all } x \in C \}.$$ 

The basic result is the Polar Cone Theorem which says that if $C$ is itself a non-empty, closed, pointed, convex cone then $(C^\ast)^\ast = C$. Before proceeding to the examples, we state and prove a corollary to that theorem.

**Corollary**

Let $A$ be an $m \times n$ real matrix and define $C = \{ x \in \mathbb{R}^n \mid Ax \leq 0 \}$. Then $C^\ast = \{ z \in \mathbb{R}^n \mid z = A^\top y, y \geq 0 \}$.

**Proof:** Let $K = \{ z \in \mathbb{R}^n \mid z = A^\top y, y \geq 0 \}$. We wish to show that $K = C^\ast$. The strategy is to show that $C = K^\ast$ and then to apply the Polar Cone Theorem. Now $x \in K^\ast$ if and only if $\langle y, Ax \rangle \leq 0$ with $y \geq 0$. This latter inequality holds if and only if the vector $Ax \leq 0$ for if one component, say the $j^{th}$ were positive, then the inner product with the unit vector $e_j$ would be positive. This shows that $C = K^\ast$ and the proof is completed by using the Polar Cone Theorem $C^\ast = (K^\ast)^\ast = K$. $lacksquare$

Section 2: Examples

In two dimensions, it is easy to draw pictures. Since $\langle y, x \rangle = \|x\| \|y\| \cos (\theta)$ the angle between vectors in the cone and in the polar cone must lie between $\pi/2$ and $3\pi/2$.

Here is a list of cones, all of which should be familiar. In each case, we can compute the polar cone.

1. A linear subspace $L \subset \mathbb{R}^n$ is a cone.
2. The set of all “non-negative vectors” in $\mathbb{R}^n$, i.e., those $x$ for which $x_i \geq 0$ for $i = 1, \ldots, n$.
3. For any $b \in \mathbb{R}^n$, the set of all vectors of the form $\lambda b$, $\lambda \geq 0$, which is called a half-line $h^+$. So $h^+ = \{ x \in \mathbb{R}^n \mid x = \lambda b, \lambda \geq 0 \}$.
4. The set of all solutions of the inequality $\langle b, x \rangle \leq 0$. This is the half-space $H^- = \{ x \in \mathbb{R}^n \mid \langle x, b \rangle \leq 0 \}$.
5. The set of all solutions of the homogeneous matrix inequality $Ax \leq 0$. 

We have checked in lecture that all the above, with the single exception of Example 3, form convex cones.

Now we identify the polar cone for each of the listed examples.

1*. If $L$ is the subspace, consider the subspace $L^\perp$, the orthogonal compliment of $L$ in $\mathbb{R}^n$ which is given by

$$L^\perp = \{ y \in \mathbb{R}^n \mid \langle y, x \rangle = 0, \text{ for all } x \in L \}.$$

Now if $y \in L^\perp$ then $\langle y, x \rangle = 0$ and so $y$ belongs to the dual cone $L^*$. Conversely, suppose $y \in L^*$ then $\langle y, x \rangle \leq 0$ for all $x \in L$. But $L$ is a subspace, so for any $x \in L$, we have also $-x \in L$. So we must also have $\langle y, -x \rangle \leq 0$ for all $x \in L$. Hence $\langle y, x \rangle \geq 0$ and so $\langle x, y \rangle = 0$. It follows that $y \in L^\perp$. Since we now have both $L^* \subset L^\perp$ and $L^\perp \subset L^*$ the two sets are equal.

2*. Denoting the non-negative orthant by $\mathbb{R}^n_\geq$ and the non-positive orthant by $\mathbb{R}^n_\leq$ we have $(\mathbb{R}^n_\geq)^* = \mathbb{R}^n_\leq$. Indeed, from the definition $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ we see immediately that $\langle x, y \rangle \leq 0$ whenever $x_i \geq 0$ and $y_i \leq 0$ for all $i = 1, \ldots, n$. Moreover, since the usual unit vectors $e_i \in \mathbb{R}^n_\geq$ no vector in $(\mathbb{R}^n_\geq)^*$ can have a positive component. So for $h^+ = \{ x \in \mathbb{R}^n \mid x = \lambda b, \lambda \geq 0 \}$, we have $y \in (h^+)^*$ if and only if $\langle y, \lambda b \rangle \leq 0$ for all $\lambda \geq 0$. But since $\langle y, \lambda b \rangle = \lambda \langle y, b \rangle$, this is equivalent to requiring $\langle y, b \rangle \leq 0$. Thus $y \in (h^+)^*$ if and only if $y \in H^-$ and so, $(h^+)^* = H^-$, the half-space in Example 4.

4*. From the reasoning in 3*, $(H^-)^* = h^+$.

5*. The matrix inequality can be rewritten as a set of inequalities for inner products $\langle x, a^j \rangle \leq 0$ where $a^j$ is the $j^{th}$ row of the matrix $A$, $j = 1, \ldots, m$. Denoting the solution set, i.e., the set of vector $x$ which satisfy these $m$ inequalities by $C$ then for any non-negative vector $\lambda \in \mathbb{R}^m$ the vector $y = \sum_{i=1}^{m} \lambda_i a^j$ satisfies $\langle x, y \rangle \leq 0$ and so belongs to $C^*$. The corollary above establishes the converse statement that every vector in $C^*$ is a non-negative linear combination of the vectors $a^j$.

Section 3: Some Useful Properties

Here is a listing of some properties that relate cones, and particularly Minkowski sums, to their polars. We assume that all cones below are pointed.

1. If $C_1 \subset C_2$ then $C_2^* \subset C_1^*$.
2. $(C_1 + C_2)^* = C_1^* \cap C_2^*$.
3. $C_1^* + C_2^* \subset (C_1 \cap C_2)^*$. 

4. $C \subset C^*$. 

It is easy to check Statement 1. indeed, if $y \in C_2^*$ then $\langle x, y \rangle \leq 0$ for all $x \in C_2$. But this latter statement implies that $\langle x, y \rangle \leq 0$ for all $x \in C_1$ since $C_1 \subset C_2$. Hence $y \in C_1^*$. ■

To verify Statement 2, suppose first that $x \in C_1^* \cap C_2^*$. Then $\langle x, y \rangle \leq 0$ for all $y^1 \in C_1$ and $\langle x, y^2 \rangle \leq 0$ for all $y^2 \in C_2$. Now, if $y \in C_1 + C_2$ then we can write $y = y^1 + y^2$ where $y^1 \in C_1$ and $y^2 \in C_2$. But then

$$\langle x, y \rangle = \langle x, y^1 \rangle + \langle x, y^2 \rangle \leq 0,$$

and so $x \in (C_1 + C_2)^*$. 

Conversely, if we start with an arbitrary $x \in (C_1 + C_2)^*$, then for every $y = y^1 + y^2$ with $y^1 \in C_1$ and $y^2 \in C_2$, we have $\langle x, y \rangle \leq 0$. In particular, if $y^2 = 0$ then this implies that $\langle x, y^1 \rangle \leq 0$ for all $y^1 \in C_1$ and, likewise, if $y^1 = 0$ then $\langle x, y^2 \rangle \leq 0$ for all $y^2 \in C_2$. Therefore $x \in C_1^* \cap C_2^*$. ■

Statement 3 can be verified similarly. Indeed, if $x \in C_1^* + C_2^*$ then we need to show that $\langle xy \rangle \leq 0$ for all $y \in C_1 \cap C_2$. Now $x = x^1 + x^2$ with $x^1 \in C_1^*$ and $x^2 \in C_2^*$. It follows that for $y \in C_1 \cap C_2$

$$\langle x, y \rangle = \langle x^1, y \rangle + \langle y, x^2 \rangle \leq 0,$$

since $y \in C_1$ and $y \in C_2$. This proves the required inclusion. ■

Finally, Statement 4 is established by simply noting that if $x \in C$ then, for all $y \in C^*$, $\langle x, y \rangle \leq 0$ which implies that $x \in (C^*)^*$. ■

Note that the Polar Cone Theorem gives conditions under which we have equality in Statement 4, rather than inclusion, but the conditions are certainly necessary for the result. Moreover, the inclusion in Statement 3 cannot be improved in general as examples show.