Chapter 4

The Farkas-Minkowski Theorem and Applications

4.1 Introduction

4.2 The Farkas-Minkowski Theorem

The results presented below, the first of which appeared in 1902, are concerned with the existence of non-negative solutions of the linear system

\[ A \mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0, \]

where \( A \) is an \( m \times n \) matrix with real entries, \( \mathbf{x} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m \). Here is a basic statement of the theorem which is due to Farkas and Minkowski.

**Theorem 4.2.1** A necessary and sufficient condition that (4.1-4.2) has a solution is that, for all \( \mathbf{y} \in \mathbb{R}^m \) with the property that \( A^\top \mathbf{y} \geq 0 \), we have \( \langle \mathbf{b}, \mathbf{y} \rangle \geq 0 \).

This theorem may be reformulated as an alternative theorem, in other words, as a theorem asserting that one set of equalities and/or inequalities has a solution if and only if another set does not. It is easy to see that the following statement is equivalent to the first:
Theorem 4.2.2 The system (4.1-4.2) has a solution if and only if the system

\[
\begin{align*}
A^\top y & \geq 0, \\
\langle b, y \rangle & < 0,
\end{align*}
\]

has no solution.

There are a number of ways to prove Theorem 4.2.1. One way is to use the duality theorem of linear programming. Since the Farkas-Minkowski Theorem is used in some discussions of linear programming, it is useful to have an independent proof even if it may be less elementary in the sense that it uses a separation theorem. This is the proof which we will present below. Once established, it can then be used to prove the Duality Theorem of Linear Programming.

Before starting, it is useful to recall some facts about cones in \( \mathbb{R}^n \).

The definition of a cone with vertex 0 shows that it is a collection of half-lines emanating from the origin. The origin itself may, or may not be in the cone. It is an elementary fact, as we have seen, that an arbitrary intersection of cones is again a cone.

Now, let \( \mathbf{x}^1, \ldots, \mathbf{x}^k \) be any \( k \) elements of \( \mathbb{R}^n \). We look at all vectors of the form \( \sum_{i=1}^{k} \mu_i \mathbf{x}^i \) where for each \( i, \mu_i > 0 \). This set is clearly a cone (in fact it is even convex) and is called the cone generated by the vectors \( \mathbf{x}^i, i = 1, \ldots, k \). It is useful to recast the definition of this particular cone as follows. Take \( A \) to be an \( n \times k \) matrix whose columns are the vectors \( \mathbf{x}^i \). Then the cone generated by these vectors is the set \( \{ z \in \mathbb{R}^n | z = A\mu, \mu \geq 0 \} \).

Before proving the theorem, we need to recall an earlier result, namely Lemma ?? regarding this particular cone which is generated by the columns of \( A \). We restate this Lemma with different notation suitable for our present purposes.

**Lemma 4.2.3** Let \( A \) be an \( m \times n \) matrix. Then the set \( \mathcal{R} = \{ z \in \mathbb{R}^m | z = Ax, x \geq 0 \} \) is a closed subset of \( \mathbb{R}^m \).

%qed Having this lemma in hand, we may turn to the proof of Theorem 4.2.1.
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Proof: First, it is easy to see that the condition is necessary. Indeed, if the system (4.1-4.2) has a non-negative solution $\mathbf{x} \geq 0$, then, for all $\mathbf{y} \in \mathbb{R}^m$ such that $A^\top \mathbf{y} \geq 0$, we have\(^1\)

$$\langle \mathbf{y}, \mathbf{b} \rangle = \langle \mathbf{y}, A\mathbf{x} \rangle = \langle A^\top \mathbf{y}, \mathbf{x} \rangle \geq 0,$$

since all terms in the inner product are products of non-negative real numbers.

To see that the condition is sufficient we assume that the system (4.1-4.2) has no solution and show that there is some vector $\mathbf{y}$ such that $A^\top \mathbf{y} \geq 0$ and $\langle \mathbf{b}, \mathbf{y} \rangle < 0$.

In order to do this, we will apply the basic separation theorem. Consider the set

$$\mathcal{R} := \{ \mathbf{z} \in \mathbb{R}^m \mid \mathbf{z} = A\mathbf{x}, \mathbf{x} \geq 0 \}.$$

Clearly this set is convex and, by the preceeding lemma, it is closed. To say that the system (4.1-4.2) has no solution says that $\mathbf{b} \notin \mathcal{R}$. Observe that the set \{ $\mathbf{b}$ \} is closed, bounded and convex. Hence, by the strict separation theorem, there exists a vector $\mathbf{a} \in \mathbb{R}^m, \mathbf{a} \neq 0$ and a scalar $\alpha$, such that

$$\langle \mathbf{a}, \mathbf{y} \rangle < \alpha \leq \langle \mathbf{a}, \mathbf{b} \rangle, \text{ for all } \mathbf{y} \in \mathcal{R}.$$

Since $0 \in \mathcal{R}$ we must have $\alpha > 0$. Hence $\langle \mathbf{a}, \mathbf{b} \rangle > 0$. Likewise, $\langle \mathbf{a}, A\mathbf{x} \rangle \leq \alpha$ for all $\mathbf{x} \geq 0$. From this it follows that $A^\top \mathbf{a} \leq 0$. Indeed, if the vector $\mathbf{w} = A^\top \mathbf{a}$ were to have a positive component, say $w_j$, then we can take $\mathbf{x} = (0, 0, \ldots, 0, M, 0, \ldots, 0)^\top$ where $M > 0$ appears in the $j^{th}$ position. Then certainly $\mathbf{x} \geq 0$ and

$$\langle A^\top \mathbf{a}, \mathbf{x} \rangle = w_j M,$$

which can be made as large as desired by choosing $M$ sufficiently large. In particular, if we choose $M > \alpha/w_j$ then the bound $\langle \mathbf{a}, A\mathbf{x} \rangle \leq \alpha$ is violated. This shows that $A^\top \mathbf{a} \leq 0$ and completes the proof. Indeed, we simply set $\mathbf{y} = -\mathbf{a}$ to get the required result. \(\blacksquare\)

There are a number of variants each of which is just a reworking of the basic theorem. Two of them are particularly useful:

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\(^1\)Recall that, for the Euclidean inner product $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^\top \mathbf{y} \rangle$. Indeed $\langle \mathbf{y}, A\mathbf{x} \rangle = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,k}x_ky_i = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{i,k}y_i \right) = \langle A^\top \mathbf{y}, \mathbf{x} \rangle$. 
(a) The system

\[
Ax \leq b, \\
x \geq 0,
\]

has a solution if and only if, for all \(y \geq 0\) such that \(A^T y \geq 0\) we have \(\langle y, b \rangle \geq 0\).

(b) The system

\[
Ax \leq b,
\]

has a solution if and only if for all \(y \geq 0\) such that \(Ay = 0\) the inner product \(\langle y, b \rangle \geq 0\).

There are also a number of closely related results. Here is one.

**Theorem 4.2.4** (Gordon) Let \(A\) be an \(m \times n\) real matrix, \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\). Then one and only one of the following conditions holds:

1. There exists and \(x \in \mathbb{R}^n\) such that \(Ax < 0\);

2. There exists a \(y \in \mathbb{R}^m, y \neq 0\) such that \(A^T y = 0\) and \(y \geq 0\).

**Proof:** Let \(\hat{e} = (1, 1, \ldots, 1)^T \in \mathbb{R}^m\). Then the first condition is equivalent to saying that \(Ax \leq -\hat{e}\) has a solution. By Theorem 4.2.1 this is equivalent to the statement that if \(y \geq 0\) and \(A^T y = 0\) then \(\langle -y, \hat{e} \rangle \geq 0\). Hence there is no \(y \neq 0\) such that \(A^T y = 0\) and \(y \geq 0\).

Conversely, if there is a \(y \neq 0\) such that \(A^T y = 0\) and \(y \geq 0\), then the condition of the Farkas-Minkowski Theorem does not hold and hence \(Ax < 0\) has no solution. □

We now turn to a basic solvability theorem from Linear Algebra which is sometimes called the **Fredholm Alternative Theorem**. While the usual proofs of this result do not use the Farkas-Minkowski result, we do so here.

**Theorem 4.2.5** Either the system of equations \(Ax = b\) has a solution, or else there is a vector \(y\) such that \(A^T y = 0\) and \(y^T b \neq 0\).
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Proof: In order to use the Farkas-Minkowski Theorem, we need to rewrite the equation \( Ax = b \) which has no sign constraint, as an equivalent system which does have such a constraint. We do this by introducing two new variables \( u \geq 0 \) and \( v \geq 0 \) with \( x + v = u \). The equation then is \( A(u - v) = b \) or, in partitioned form

\[
[A, -A] \begin{pmatrix} u \\ v \end{pmatrix} = b,
\]

with non-negative solution \((u, v)^\top \geq 0\). The Farkas alternative is then that \( y^\top [A, -A] \geq 0 \), \( y^\top b < 0 \) has a solution \( y \). These inequalities mean, in particular, that \( y^\top A \geq 0 \) and \( -y^\top A \geq 0 \) which together imply that \( y^\top A = 0 \). So the Farkas alternative says \( y^\top A = 0 \), \( y^\top b < 0 \) has a solution.

Without loss of generality, we may replace \( y \) by \(-y\) so that we now have \( y^\top A = 0 \) and \( y^\top b \neq 0 \). This is exactly the Fredholm Alternative. ■

The Fredholm Alternative is often stated in a slightly different way which leads some people to say that “uniqueness implies existence”, probably not a good way to think of the result.

Corollary 4.2.6 If the homogeneous equation \( y^\top A = 0 \) has a unique solution then there exists a solution to the system \( Ax = b \).

Let us illustrate the application of the Minkowski-Farkas result in two simple cases.

Example 4.2.7 Does the algebraic system

\[
\begin{pmatrix} 4 & 1 & -5 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

have a non-negative solution?

The Farkas Alternative for this system is the dot product \((1, 1), (y_1, y_2)\) < 0 together with

\[
\begin{pmatrix} 4 & 1 \\ 1 & 0 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
Since $y_1 \geq 0$ from the second row, the dot product $\langle B, y \rangle < 0$ requires that $y_2 < 0$. But then the inequality from the last row is violated. This shows that there are no solutions to the alternative system, hence the original system has a non-negative solution.

**Example 4.2.8** We look for non-negative solutions of the equation

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
2 \\
2 \\
2 \\
1
\end{pmatrix}.
$$

The Farkas Alternative is

$$
\begin{pmatrix}
1 & 0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix}
\geq
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

together with

$$2y_1 + 2y_2 + 2y_3 + y_4 < 0.$$ 

Since this system has the solution $y_1 = y_2 = y_3 = -1/2$ and $y_4 = 1$, there are no non-negative solutions of the original equation.

Finally, we look at an application of the Fredholm Alternative.

**Example 4.2.9** Consider the set of equations

$$
\begin{align*}
2x + 3y &= 1 \\
x - 3y &= 1 \\
-x + y &= 0
\end{align*}
$$

Now look at the transposed system


\[
\begin{align*}
2u + v - w &= 0 \\
3u - v + w &= 0
\end{align*}
\]

together with \( u + v \neq 0 \).

Since this is a homogeneous system, we may well normalize this last requirement to \( u + v = 1 \). Then look at the matrix

\[
\begin{pmatrix}
2 & 1 & -1 \\
3 & -3 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

It is easy to check that this matrix has row-eschelon form

\[
\begin{pmatrix}
2 & 1 & -1 & 0 \\
0 & -9/5 & 5/2 & 0 \\
0 & 0 & 7/9 & 1
\end{pmatrix}
\]

and hence the non-homogeneous system has a non-trivial solution. Hence the original system has no solution.

4.3 Applications to Linear Programming

As we mentioned in the beginning, the Farkas-Minkowski Alternative is often proved, not by recourse to a separation theorem but from the Duality Theorem of Linear Programming. However, the Farkas-Minkowski theorem leads to a particularly simple proof of the Duality Theorem. Let us explain.

Linear programming deals with problems of the form

\[
\begin{align*}
\text{minimize} & \quad \langle c, x \rangle \\
\text{subject to} & \quad Ax = b, \\
& \quad x \geq 0.
\end{align*}
\]

Where \( x, c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \) and \( A \) is an \( m \times n \) matrix.

This problem is called the primal problem. The dual problem to this primal is...
maximize \( \langle b, y \rangle \)
subject to \( A^\top y \leq c \).

It is clear that the feasible region, that is the set of points that satisfy the constraints in either program is a convex set. These convex sets are called the feasible sets for the two problems. The first thing that we can say concerning the feasible points, while simple, is of great use.

**Theorem 4.3.1** *(The Weak Duality Theorem)* Let \( x \) be a feasible point for the primal problem, and \( y \) a feasible point for its dual, then \( \langle c, x \rangle \geq \langle b, y \rangle \).

**Proof:**

\[
\langle c, x \rangle - \langle b, y \rangle = \langle c, x \rangle - \langle Ax, y \rangle = \langle x, c \rangle - \langle x, A^\top y \rangle \\
= \langle x, c \rangle - \langle x, A^\top y \rangle = \langle x, c - A^\top y \rangle \geq 0,
\]

since, both arguments in the last inner product are vectors with positive components. Hence we have the inequality \( \langle c, x \rangle \geq \langle b, y \rangle \). 

We call the value \( \langle c, x \rangle - \langle b, y \rangle \) the **duality gap**.

This theorem shows that a feasible solution to either problem yields a bound on the value of the other. From that fact we can say a number of things. First, suppose that the primal problem is unbounded, i.e. \( \inf \langle c, x \rangle = -\infty \). Then the set of feasible points for the dual must be empty since, if there were a feasible point \( y \) then \( \langle b, y \rangle \) would be a lower bound for \( \langle c, x \rangle \) which is a contradiction. Similarly, if the dual is unbounded, then the primal problem is infeasible.

Also, if we have feasible solution for both problems for which the duality gap is zero, then these feasible points are optimal solutions for their respective problems. Indeed, since a dual feasible solution gives an upper bound to the cost functional of the primal, if they are equal then the primal has acheived its maximum value. Likewise for the dual problem.

We are now in a position to state and prove the **strong duality property** which, when established, shows that we may solve the dual problem in place of the primal and this will lead to a solution of the primal itself.
Theorem 4.3.2 (Strong Duality) Suppose that both the primal and the dual problems have feasible solutions. Then $\bar{x}$ is optimal for the primal problem if and only if

(a) $\bar{x}$ is feasible for the primal.

(b) there is a feasible point for the dual, $\bar{y}$, that satisfies

$$\langle c, \bar{x} \rangle = \langle b, \bar{y} \rangle$$

Proof: Since, by weak duality, we always have $\langle c, x \rangle \geq \langle b, y \rangle$ for all feasible points $x$ for the primal and $y$ for the dual, we need to show that there is a feasible primal/dual pair $\{\bar{x}, \bar{y}\}$ for which the reverse inequality is true.

Let $\epsilon > 0$ and consider the system

$$-\langle c, x \rangle \leq -\langle c, \bar{x} \rangle - \epsilon$$

$$Ax = b$$

$$x \geq 0$$

By definition of optimality for the primal problem, this system has no solution. Hence, by the Farkas-Minkowski Theorem the system

$$-\lambda c + A^T y \geq 0$$

$$\lambda(-\langle c, \bar{x} \rangle - \epsilon) + \langle y, b \rangle < 0$$

$$\lambda \geq 0,$$ 

has a solution. Denote that solution by $(\lambda^*, y^*)$. Now, $\lambda^* \geq 0$, but if equality were to hold, then the last system of inequalities would reduce to

$$A^T y^* \geq 0$$

$$\langle y^*, b \rangle < 0.$$ 

But then the Farkas alternative implies that the system $A x = 0$ has no non-negative solution. Hence the primal problem is infeasible which violates the original assumption. Hence we must have $\lambda^* > 0$. 

Now, we define \( \lambda' = \frac{y^*}{\lambda^*} \) and note that
\[
A^\top y' = \frac{A^\top y^*}{\lambda^*} \geq c
\]
which means that \( y' \) is a feasible point for the dual problem. Moreover \( \langle y', b \rangle < \langle c, \bar{x} \rangle - \epsilon \) as a simple computation shows. Since \( y' \) is feasible for the dual problem, we then have
\[
\langle c, \bar{x} \rangle \leq \langle y^*, b \rangle \leq \langle y', b \rangle < \langle c, \bar{x} \rangle + \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, it follows that the optimal values of the primal and dual are the same: \( \langle c, \bar{x} \rangle = \langle y^*, b \rangle \).

### 4.4 De Finetti’s Arbitrage Theorem

Arbitrage is the practice of buying one commodity at a low price in one market and making a risk-free profit by taking advantage of a higher price in another market. As a trivial example, suppose that the market near your home is selling bananas for $0.29/pound while the market near your mother-in-law’s house is selling them for $0.40/pound. This price differential can support a more harmonious family, since you are motivated to buy, say 10 pounds of bananas at your local market, and sell them to the market near the in-laws. Hence your purchase price is $2.90 and your selling price is $4.00 for a profit of $1.10.

You can imagine doing this often and continuing, at least for a while, to make the profit. However, the elementary laws of supply and demand will tell us that if you make this trade often enough, the supply of bananas will decrease at your neighborhood market and the price will go up, while the price at the distant store, where the supply will increase, will tend to decrease. Eventually a price equilibrium will be established and the market in bananas will be “arbitrage free”.

The first mathematical result was given by Augustin Cournot in 1838 in his book *Researches on the Mathematical Principles of the Theory of Wealth*. In that book he stated what might be called the first arbitrage theorem:

There exists a system of absolute prices for commodities such that the exchange rates (or relative prices) are price ratios if and only if the exchange rates are arbitrage-free.
In these notes, we want to look at the closely related statement due to De Finetti. To set the stage, suppose that there is an experiment having $m$ possible outcomes for which there are $n$ possible wagers. That is, if you bet the amount $y$ on wager $i$ you win the amount $y r_i(j)$ if the outcome of the experiment is $j$. Here $y$ can be positive, negative, or zero. A betting strategy is a vector $y = (y_1, \ldots, y_n)\top$ which means that you simultaneously bet the amount $y_i$ on wager $i$ for $i = 1, \ldots, n$. So if the outcome of the experiment is $j$, your gain (or loss) from the strategy $y$ is given by

$$\sum_{j=1}^{n} y_j r_j(j).$$

The result of De Finetti is the following:

**Theorem 4.4.1** Exactly one of the following is true: Either

(a) there exists a probability vector $p = (p_1, \ldots, p_m)$ with

$$\sum_{j=1}^{m} r_i(j) p_j = 0,$$

or

(b) there exists a betting strategy $y$ such that

$$\sum_{i=1}^{n} y_i r_i(j) > 0, \text{ for all } j = 1, 2, \ldots, m.$$

In other words, either there exists a probability distribution on the outcome under which all bets have expected gain equal to zero, or else there is a betting strategy which always results in a positive win.

**Proof:** Consider the matrix

$$A := \begin{pmatrix} r_1(1) & \cdots & r_1(m) \\ \vdots & \ddots & \vdots \\ r_n(1) & \cdots & r_n(m) \\ -1 & \cdots & -1 \end{pmatrix}.$$
and the vector $b := (0, 0, \ldots, 0, -1)^\top$. Then consider the system

$$Ap = b, \ p \geq 0, p = (p_1, \ldots, p_m)^\top,$$

which reads

$$\sum_{j=1}^{m} r_i(j) p_j = 0, \ i = 1, \ldots, n$$

while the last row, the $n + 1\text{st}$ row reads

$$\sum_{j=1}^{m} -p_j = -1.$$

Now the Farkas-Minkowski Theorem tells us that either this system has a solution $p$ or the system

$$y^\top A \geq 0, \ y^\top b < 0$$

has a solution $y \in \mathbb{R}^n$ but not both.

But with our choice of matrix $A$, the inequality $y^\top A \geq 0$ reads (note that $A$ is an $m \times (n + 1)$-matrix)

$$\sum_{i=1}^{n} y_i r_i(j) - y_{n+1} \geq 0, \ j = 1, \ldots, m,$$

while the second inequality $y^\top b < 0$ reads just $-y_{n+1} < 0$. Thus we have

$$\sum_{i=1}^{n} y_i r_i(j) \geq y_{n+1} > 0, \ j = 1, \ldots, m.$$

This is the second possibility of the theorem.

4.5 European Call Options and Fair Prices

A call option is a contract between two parties, the seller and the buyer, that the buyer has the right to buy a certain commodity or certain financial instrument from the seller at a certain agreed upon price at some time in the future. The buyer has no obligation to buy, but the seller does have the obligation to sell at the agreed upon price (usually called
the strike price) if the call is exercised. In exchange for this obligation, the buyer usually pays a premium or option price as the price of the option when it is originally purchased.

The two parties are making two different bets; the seller is betting that the future price of the commodity is less than the strike price. In this case, the seller collects the premium and does not receive any gain if the commodity prices rises above the strike price. The buyer is betting that the future price will be higher than the strike price, so that he can buy the commodity at a price lower than the prevailing one and then sell, thereby making a profit. While the buyer may make a great deal of profit, his loss is limited by the premium that is paid.

The most common types of options are the American Option and the European Option. The former allows the option to be exercised at any time during the life of the option while the latter allows it to be exercised only on the expiration date of the option. Here, we are going to look at a simple example of a one period European call option and ask what a “fair” price for the option is.

We assume that there are only two possible outcomes, so that \( m = 2 \) and the matrix \( A \) is given by

\[
A = \begin{pmatrix} r(1) & r(2) \\ -1 & -1 \end{pmatrix}.
\]

If \( r(1) \neq r(2) \) then the matrix \( A \) is non-singular and has inverse

\[
A^{-1} = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix},
\]

so that, with \( b \) as in the preceding section \( b = (0, -1)^T \) and

\[
A^{-1} b = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.
\]

So if \( r(2) > 0 \) and \( r(1) < 0 \) then the right-hand side of the above relation can be interpreted as a probability vector

\[
p = \begin{pmatrix} 1 - p \\ p \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.
\]
Of course, if both are positive, any wager which assigns a positive bet to both is a guaranteed win, and if both are negative, any wager that assigns a negative bet to both is a guaranteed win.

Now suppose that we allow another wager. This new wager has the return $\alpha$ if the outcome is 1 and $\beta$ if the outcome is 2. Then, according to De Finetti’s theorem, unless

$$\alpha (1 - p) + \beta p = 0,$$

there will be some combination of the two wagers that has a guaranteed profit.

we now assume that there is a certain stock which at time $t = 0$ has the price $S(0)$. Suppose that at time $t = 1$ there are two possibilities, either the value of the stock can go up by a fixed factor $u$ so that the price at $t = 1$ is $S(1) = u S(0)$ or it can go down by a fixed factor $d$ so that $S(1) = d S(0)$, $u > 1 > d$. Moreover, suppose that the interest rate on a deposit in a bank for this period is $r$ so that the current value of $M$ is $M(1 + r)^{-1}$.

So the profit per dollar if the stock goes up is $r(2) = (u/r + 1) - 1$, while if the stock goes down it is $r(1) = (d/1 + r) - 1$. In terms of the probability, $p$, defined above, we have

$$1 - p = \frac{u - 1 - r}{u - d},$$

$$p = \frac{1 + r - d}{u - d}.$$

We now want to decide what is a “fair price” for a European call option. Let $K$ be the strike price at the end of one time period. Let $C$ be the current price of the call option. Define the parameter $k$ by

$$K = k S(0).$$

Suppose for the moment that the stock goes up. Since you can buy the stock at time $t = 1$ for a price of $k S(0)$ (the strike price) and then sell it immediately at the price $u S(0)$, and since the price of the option today is $\$C$, the gain per unit purchased is

$$\frac{u - k}{1 + r} - C.$$
Of course, if the stock goes down by a factor of \( d \), you lose \$C^2\.

Now, De Finetti’s theorem says that, unless

\[
0 = -(1 - p) C + p \left( \frac{u - k}{1 + r} - C \right) = p \cdot \frac{u - k}{1 + r} - C
\]

\[
= \frac{1 + r - d}{u - d} \cdot \frac{u - k}{1 + r} - C,
\]

there is a mixed strategy of buying or selling the stock and buying and selling the option with a sure profit. Thus the price of the option should be set at the “fair price” given by

\[
C = \frac{1 + r - d}{u - d} \cdot \frac{u - k}{1 + r}.
\]

This is the fair price of the option in the sense that if the option were priced differently, an arbitrageur could make a guaranteed profit.

\[^2\text{This is under the assumption that } d \leq k \leq u.\]