1. (25 points) Let $u = \hat{i} + 2\hat{j} - \hat{k}$, $v = -2\hat{i} - 4\hat{j}$, and $w = 7\hat{j} - 4\hat{k}$.

(a) Find the projection of $w$ onto $u \times v$.

**A:**

$$(u \times v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & -4 & 0 \end{vmatrix} = -4\hat{i} + 2\hat{j}$$

Computing the norm of $u \times v$ yields

$$\sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}.$$  

Taking the dot product of $w$ with $u \times v$ gives

$$\begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} = 7 \cdot 2 = 14.$$  

The projection of $w$ on $v \times v$ is then

$$\text{proj}_{u \times v} w = \left( \frac{w \cdot u \times v}{(u \times v) \cdot (u \times v)} \right) w = \left( \frac{7}{5\sqrt{5}} \right) (-2, 1, 0)^\top.$$  

(b) Compute the volume of the parallelepiped determined by the vectors $u, v,$ and $w$.

**A:** The volume of the parallelepiped is just the norm of the triple scalar product $(u \times v) \cdot w$. But we have computed this above. It is 14.

2. (25 points) Consider the plane whose equation is $x + 3y + z = 5$.

(a) Find a unit normal to the plane containing the three points $P : (1, 1, 7)$, $Q : (3, 1, 5)$, and $R : (2, 0, 3)$.  

**A:** The vectors $\overrightarrow{PQ} = (2, 0, -2)^\top$ and $\overrightarrow{PR} = (1, -1, -4)^\top$ lie in the plane, so a normal vector is given by the cross product:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ 1 & -1 & -4 \end{vmatrix} = (-2, 6, -2)^\top = n.$$  

This vector $n$ is normal to the plane. A unit normal $\hat{n}$ is given by

$$\frac{n}{\|n\|} = \frac{1}{\sqrt{4 + 36 + 4}} (-2, 6, -2)^\top = \frac{1}{\sqrt{11}} (-1, 3, -1)^\top.$$
(b) Write the equation of the plane of part (b) in the form \((\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0\). Rewrite this equation as a single equation involving the components \(x, y, \) and \(z\) of the vector \(\mathbf{x}\).

\[ \begin{align*}
\mathbf{x} &= (x, y, z)^\top, \\
\mathbf{p} &= (1, 1, 7)^\top, \quad \text{and} \\
\mathbf{n} &= (-2, 6, -2)^\top.
\end{align*} \]

Then
\[
\begin{pmatrix}
(x - 1) \\
(y - 1) \\
(z - 7)
\end{pmatrix} \cdot 
\begin{pmatrix}
-2 \\
6 \\
-2
\end{pmatrix} = 0
\]

is the required equation. Computing the dot product yields the equation 
\[-2(x - 1) + 6(y - 1) - 2(z - 7) = 0\] or 
\[-2x + 6y - 2z = -10.\]

(c) Find a vector perpendicular to the vector \(\mathbf{q} = (3, 1, 5)^\top\) and which lies in the plane or part b.

\[
\begin{align*}
\mathbf{v} \cdot \mathbf{q} &= 0. \\
\text{That it lie in the plane means that it satisfy as well} \\
\mathbf{n} \cdot \mathbf{v} &= 0.
\end{align*}
\]

This gives two equations
\[
\begin{align*}
3x + y + 5z &= 0 \\
-2x + 6y - 2z &= 0.
\end{align*}
\]

These equations have the solution \(x = 1, y = -\frac{49}{8}, z = \frac{5}{8}\). So the vector with these components satisfies the conditions.

3. (25 points)

(a) Find a parametric equation for the line passing through the point \(P : (1, 2, 3)\) and parallel to the vector \(\mathbf{v}\) in part (a).

\[
\begin{align*}
\mathbf{x} &= (x, y, z)^\top, \\
\mathbf{p} &= (1, 1, 7)^\top, \quad \text{and} \\
\mathbf{n} &= (-2, 6, -2)^\top.
\end{align*} \]

Then
\[
\begin{pmatrix}
(x - 1) \\
(y - 1) \\
(z - 7)
\end{pmatrix} \cdot 
\begin{pmatrix}
-2 \\
6 \\
-2
\end{pmatrix} = 0
\]

is the required equation. Computing the dot product yields the equation 
\[-2(x - 1) + 6(y - 1) - 2(z - 7) = 0\] or 
\[-2x + 6y - 2z = -10.\]

(b) Find the perpendicular bisector of the line segment joining the points \(P : (1, 2, 3)\) and \(Q : (4, 2, -4)\).

\[
\begin{align*}
\mathbf{x} &= (x, y, z)^\top, \\
\mathbf{p} &= (1, 1, 7)^\top, \quad \text{and} \\
\mathbf{n} &= (-2, 6, -2)^\top.
\end{align*} \]

Then
\[
\begin{pmatrix}
(x - 1) \\
(y - 1) \\
(z - 7)
\end{pmatrix} \cdot 
\begin{pmatrix}
-2 \\
6 \\
-2
\end{pmatrix} = 0
\]

is the required equation. Computing the dot product yields the equation 
\[-2(x - 1) + 6(y - 1) - 2(z - 7) = 0\] or 
\[-2x + 6y - 2z = -10.\]

These equations have the solution \(x = 1, y = -\frac{49}{8}, z = \frac{5}{8}\). So the vector with these components satisfies the conditions.
(c) If \( \mathbf{m} \) is the position vector of the midpoint and \( \mathbf{r} \) is the position vector of the point \((6, 2, 1)\), are the vectors \( \mathbf{m} - \mathbf{q} \) and \( \mathbf{v} \) linearly independent? Justify your answer.

**A:** From the preceding part, the midpoint is \( \left( \frac{5}{2}, 2, -\frac{1}{2} \right) \). Hence the vector

\[
\mathbf{m} - \mathbf{r} = \begin{pmatrix} \frac{5}{2} \\ 2 \\ -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -7 \\ 0 \\ -3 \end{pmatrix}.
\]

Now just check the dot product:

\[
(\mathbf{m} - \mathbf{r}) \cdot \mathbf{v} = \begin{pmatrix} -7 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix} = 0
\]

4. (25 points)

Let \( V \) be the vector space of points in \( \mathbb{R}^3 \).

(a) If \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) are in \( V \), define the set \( \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \). What is its dimension?

**A:** \( \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \{t\mathbf{u} + s\mathbf{v} + t\mathbf{w} \mid t, s \in \mathbb{R}\} \). This is a subspace of \( V \) of dimension three provided that the given vectors are linearly independent. If they are linearly dependent, the dimension of the subspace is either two or one.

(b) For the specific choice \( \mathbf{u} = (2, -2, 1)^\top \) and \( \mathbf{v} = (-1, 0, 1)^\top \), identify \( \text{span}\{\mathbf{u}, \mathbf{v}\} \). (Hint: derive a single equation relating \( x, y \), and \( z \) from a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \).

**A:** For these vectors we have

\[
\text{span} \{\mathbf{u}, \mathbf{v}\} = \left\{ t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}
\]

\[
= \left\{ \begin{pmatrix} 2t - s \\ -2t \\ t + s \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.
\]

Hence \( \mathbf{x}(t, s) = (x, y, z)^\top = (2t - s, -2t, t + s)^\top \). Eliminating the parameters, we have \( s = z - t \) and \( y = -\left( \frac{1}{2} \right) t \) so that \( z = -x + y \) from which it follows that \( x = y - z + y \) or \( x - y + z = 0 \). We conclude that we can write:

\[
\text{span}\{\mathbf{u}, \mathbf{v}\} = \{x - y + z = 0 \mid x, y, z \in \mathbb{R}\}.
\]