Selected Results in Combinatorics and Graph Theory

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Dissertation Defense
Outline of Talk

• Part 1: Enumerating Independent Sets of a Fixed Size in Graphs

• Part 2: Designs from Paley Graphs and Peisert Graphs

• Part 3: Sprint Relay Graphs

• Part 4: Hamiltonian Cycles in Sparse Pseudorandom Bipartite Graphs
Outline of Talk

- **Part 0**: Definitions and Notation
- **Part 1**: Enumerating Independent Sets of a Fixed Size in Graphs
- **Part 2**: Designs from Paley Graphs and Peisert Graphs
- **Part 3**: Sprint Relay Graphs
- **Part 4**: Hamiltonian Cycles in Sparse Pseudorandom Bipartite Graphs
For $t \in \mathbb{N}$, let $K_t$ denote the complete $t$-vertex graph.

$K_5 = \begin{figure}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (-1,1) {2};
\node (3) at (0,-1) {3};
\node (4) at (1,1) {4};
\node (5) at (1,-1) {5};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (4);
\draw (1) -- (5);
\draw (2) -- (3);
\draw (2) -- (4);
\draw (2) -- (5);
\draw (3) -- (4);
\draw (3) -- (5);
\draw (4) -- (5);
\end{tikzpicture}
\end{figure}$
Notation for Common Classes of graphs

For $t \in \mathbb{N}$, let $C_t$ denote the $t$-vertex cycle graph.

$C_5 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}$
For $t \in \mathbb{N}$, let $P_t$ denote the $t$-vertex path graph.

$$P_6 = \begin{array}{c}
1 \\
2 \\
3 \\
\hline
2 \\
3 \\
4 \\
\hline
5 \\
6 \\
\end{array}$$
Notation for Common Classes of Graphs

For $t_1, \ldots, t_k \in \mathbb{N}$, $K_{t_1, \ldots, t_k}$ denotes the complete multipartite graph (with partite sets of size $t_1, t_2, \ldots, t_k$).

\[ K_{3,3} = \]

\begin{tikzpicture}
  \node[shape=circle,draw=black,minimum size=1cm] (1) at (0,0) {1};
  \node[shape=circle,draw=black,minimum size=1cm] (2) at (0,-1) {2};
  \node[shape=circle,draw=black,minimum size=1cm] (3) at (0,-2) {3};
  \node[shape=circle,draw=black,minimum size=1cm] (4) at (1,0) {4};
  \node[shape=circle,draw=black,minimum size=1cm] (5) at (1,-1) {5};
  \node[shape=circle,draw=black,minimum size=1cm] (6) at (1,-2) {6};
  \draw (1)--(2)--(3);
  \draw (1)--(4)--(5)--(6);
\end{tikzpicture}
**Notation for Common Classes of Graphs**

For $t_1, \ldots, t_k \in \mathbb{N}$, $K_{t_1, \ldots, t_k}$ denotes the complete multipartite graph (with partite sets of size $t_1, t_2, \ldots, t_k$).

$$K_{3,2,2} =$$

![Diagram of a complete multipartite graph with partite sets of size 3, 2, 2]
We use an overbar to denote graph complementation.
Notation for Common Classes of Graphs

\[ \overline{K_5} = \]

1  2  3  4  5
Notation for Common Classes of Graphs

\[ C_5 \]
Notation for Common Classes of Graphs

\[ K_{3,2,2} = \]

\[ \begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\text{6} \\
\text{7} \\
\end{array} \]
NOTATION FOR COMMON CLASSES OF GRAPHS

ETC.
Notation for Common Classes of Graphs

We use \( \sqcup \) to denote disjoint union of graphs.
Selected Results in Combinatorics and Graph Theory
Definitions and Notation

**Notation for Common Classes of Graphs**

\[ K_3 \cup K_3 \cup K_2 = \]

![Graph Diagram]

1. 3
2. 2
4. 5
7. 6
8. 8
**Notation for Common Classes of Graphs**

\[ K_3 \cup K_3 \cup K_1 \cup K_1 = \]

![Graph Diagram](image_url)
NOTATION FOR COMMON CLASSES OF GRAPHS

ETC.
Let $R_{4,1}$ denote the Diamond Graph:
Let $R_{4,2}$ denote the Paw Graph:
Subgraph Notation

For any graph $G$, let

- $V(G) := \{ \text{Vertices of } G \}$ and $E(G) := \{ \text{Edges of } G \}$.
**Subgraph Notation**

For any graph $G$, let

- $V(G) := \{ \text{Vertices of } G \}$ and $E(G) := \{ \text{Edges of } G \}$.

- For $S \subseteq V(G)$, $G[S] := \text{Subgraph of } G \text{ Induced by } S$. I.e.,
  
  $V(G[S]) = S$ and $E(G[S]) = \{ xy \in E(G) : x \in S, y \in S \}$. 


Selected Results in Combinatorics and Graph Theory
Definitions and Notation

**Subgraph Notation**

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  \[
  V(G[S]) = S \quad \text{and} \quad E(G[S]) = \{ xy \in E(G) : x \in S, y \in S \}.
  \]

- For graph \( H \), \( H(G) := \{ S \subseteq V(G) : G[S] \cong H \} \).
Subgraph Notation

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- For graph $H$, $H(G) := \{S \subseteq V(G) : G[S] \cong H\}$.

- “$h(G) := |H(G)|$.”
**Subgraph Notation**

For any graph $G$, let

- $V(G) := \{ \text{Vertices of } G \}$ and $E(G) := \{ \text{Edges of } G \}$.

- For $S \subseteq V(G)$, $G[S] := \text{Subgraph of } G \text{ Induced by } S$. I.e.,
  
  $V(G[S]) = S$ and $E(G[S]) = \{ xy \in E(G) : x \in S, y \in S \}$.

- For graph $H$, $H(G) := \{ S \subseteq V(G) : G[S] \cong H \}$.

- $k_3(G) := |K_3(G)|$, $k_2 \uplus k_2(G) := |K_2 \uplus K_2(G)|$, etc.
For example, if $G$ is the graph:

\[ K_3(G) = \{ \{1, 2, 3\}, \{2, 4, 6\}, \{3, 5, 7\} \} \quad \text{and} \quad k_3(G) = 3. \]
For example, if $G$ is the graph:

![Graph Diagram]

\[ \overline{K}_3(G) = \{ \{1, 4, 5\}, \{1, 4, 7\}, \{1, 6, 5\}, \{1, 6, 7\} \} \] and \( \overline{k}_3(G) = 4 \).
For example, if $G$ is the graph:

* $K_4(G) = \overline{K_4}(G) = \emptyset$, and
* $k_4(G) = \overline{k_4}(G) = 0$. 
For $t \in \mathbb{N}$,

Elements of $\overline{K}_t(G)$ are called the \textit{Independent Sets of} $G$.

As is standard, $i_t(G) := \text{Number of Independent Sets of } G$, i.e.,

\[ i_t(G) := \overline{k}_t(G). \]
Part 1: Enumerating Independent Sets of a Fixed Size in Graphs
Part 1 is based on:

A new method for enumerating independent sets of a fixed size in general graphs with Tim Mink, *Journal of Graph Theory*, **81** (2016), no. 1, 57-72
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A new method for enumerating independent sets of a fixed size in general graphs with Tim Mink, *Journal of Graph Theory*, **81** (2016), no. 1, 57-72

arXiv:1308.3242
**Definition of the Function $\psi$**

On graph $G$, define functions $\psi_e = \psi_e(G)$ and $\psi_o = \psi_o(G)$ by
**Definition of the Function** $\psi$

On graph $G$, define functions $\psi_e = \psi_e(G)$ and $\psi_o = \psi_o(G)$ by

$$\psi_e : V(G) \rightarrow \{0, 1, 2, \ldots\}$$

$$J \mapsto \text{Number of edge-coverings of } G[J] \text{ by an even number of edges of } E(G[J])$$
Definition of the Function $\psi$

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and

$$\psi_o : V(G) \rightarrow \{0, 1, 2, \ldots\}$$

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Definition of the Function $\psi$

On graph $G$, define functions $\psi_e = \psi_e(G)$ and $\psi_o = \psi_o(G)$ by

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$$
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$$

and

$$
\psi_o : V(G) \rightarrow \{0, 1, 2, \ldots\}
$$

$$
J \mapsto \text{Number of edge-coverings of } G[J] \text{ by an odd number of edges of } E(G[J])
$$

Also, define $\psi = \psi(G)$ by

$$
\psi : V(G) \rightarrow \{0, \pm1, \pm2, \ldots\}
$$

$$
J \mapsto \psi_e(J) - \psi_o(J).
$$
Examples of $\psi$ values

Let $G$ be the graph:

Then,

$$\psi_e(\{1, 2, 6, 7\}) = 1.$$
EXAMPLES OF $\psi$ VALUES

Let $G$ be the graph:

\[
\begin{align*}
1 & \quad 2 \\
3 & \quad 4 \\
5 & \quad 6 \\
6 & \quad 7.
\end{align*}
\]

Then,

$$\psi_o(\{1, 2, 6, 7\}) = 0.$$
Examples of $\psi$ values

Let $G$ be the graph:

Then,

$$\psi(\{1, 2\}) = 0 - 1 = -1.$$
Examples of $\psi$ values

Let $G$ be the graph:

Then,

$$
\psi(\{3, 4, 5\}) = 3 - 1 = 2.
$$
Main Result

Theorem

For $n, t \in \mathbb{N}$ and any $n$-vertex graph $G$,

$$i_t(G) = \binom{n}{t} + \sum_{j=1}^{t} \binom{n-j}{t-j} \sum_{J \subseteq V(G)} \psi(J).$$
Main Result

Theorem

For \( n, t \in \mathbb{N} \) and any \( n \)-vertex graph \( G \),

\[
i_t(G) = \binom{n}{t} + \sum_{j=1}^{t} \binom{n-j}{t-j} \sum_{\substack{J \subseteq V(G) \\ |J|=j}} \psi(J).
\]

For example,

\[
i_2(G) = \binom{n}{2} + \binom{n-1}{2-1} \sum_{\substack{J \subseteq V(G) \\ J \cong K_1}} 0 + \binom{n-2}{2-2} \left( \sum_{\substack{J \subseteq V(G) \\ J \cong K_2}} -1 + \sum_{\substack{J \subseteq V(G) \\ J \cong K_2}} 0 \right)
\]

\[= \binom{n}{2} - |E(G)|.\]
We prefer the form:

\[ i_2(G) + k_2(G) = \binom{n}{2} \]
With a bit of similar simplifying, if \( d_v := \) degree of vertex \( v \) in \( G \),

\[
i_3(G) + k_3(G) = \binom{n}{3} - (n - 2)|E(G)| + \sum_{v \in V(G)} \binom{d_v}{2}
\]
With a bit of similar simplifying, if $d_v :=$ degree of vertex $v$ in $G$,

$$i_3(G) + k_3(G) = \binom{n}{3} - (n - 2)|E(G)| + \sum_{v \in V(G)} \binom{d_v}{2}$$

(which is a known formula of Goodman)
With a bit of similar simplifying, if \( d_v \) := degree of vertex \( v \) in \( G \),

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\]

(which is a known formula of Goodman) and

\[
i_4(G) - k_4(G) = \binom{n}{4} - \binom{n - 2}{2}|E(G)|
+ (n - 3) \left[ \sum_{v \in V(G)} \binom{d_v}{2} - k_3(G) \right]
+ k_2 \cup k_2(G) - c_4(G)
\]

(which was not known previously).
If $G$ is an $n$-vertex, $r$-regular graph of girth at least 5, then

$$i_3(G) = \binom{n}{3} - \frac{n(n-2)r}{2} + n\binom{r}{2}$$

and

$$i_4(G) = \binom{n}{4} - \binom{n-2}{2}\frac{nr}{2} + n(n-3)\binom{r}{2} - \binom{nr}{3} + k_2 \cup k_2(G).$$
Part 2: Designs from Paley Graphs and Peisert Graphs
Part 2 is based on:

Designs from Paley graphs and Peisert graphs, *Discrete Mathematics*, Submitted on 10/5/15

arXiv:1507.01289
Definition of the Paley Graph

For any prime power $q$ satisfying $q \equiv 1 \pmod{4}$, the $q$-vertex Paley Graph is defined as the graph with vertex set $\mathbb{F}_q$ and edge set consisting of all $\{x, y\} \subseteq \mathbb{F}_q$ so that $x \neq y$ and $x - y$ is a quadratic residue.
Definition of the Paley Graph

**Definition**

For any prime power $q$ satisfying $q \equiv 1 \pmod{4}$, the $q$-vertex *Paley Graph* is defined as the graph with vertex set $\text{GF}(q)$ and edge set consisting of all $\{x, y\} \subseteq \text{GF}(q)$ so that $x \neq y$ and $x - y$ is a quadratic residue.

Note:

- $q$ being a prime power ensures that the field $\text{GF}(q)$ exists.
- $q \equiv 1 \pmod{4}$ ensures that $x - y$ is a quadratic residue if and only if $y - x$ is a quadratic residue (and, in particular, that the undirected graph is well-defined).
Definition of the Peisert Graph

**Definition**

Let $p$ be a prime with $p \equiv 3 \pmod{4}$, let $r \in \mathbb{N}$ be even, and let $q = p^r$. The Peisert graph is defined as the graph with vertex set $GF(q)$ and edge set consisting of, for some fixed primitive root $\omega$ of $GF(q)$, all $\{x, y\} \subseteq GF(q)$ which satisfy that $x - y = \omega^j$ for $j \equiv 0, 1 \pmod{4}$. Note: $q$ being a prime power ensures that the field $GF(q)$ exists. It is straightforward to check that this construction does not depend on the choice of primitive root $\omega$. 

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DEFINITION OF THE PEISERT GRAPH

DEFINITION

Let $p$ be a prime with $p \equiv 3 \pmod{4}$, let $r \in \mathbb{N}$ be even, and let $q = p^r$. The Peisert graph is defined as the graph with vertex set $\text{GF}(q)$ and edge set consisting of, for some fixed primitive root $\omega$ of $\text{GF}(q)$, all $\{x, y\} \subseteq \text{GF}(q)$ which satisfy that $x - y = \omega^j$ for $j \equiv 0, 1 \pmod{4}$.

Note:

- $q$ being a prime power ensures that the field $\text{GF}(q)$ exists.
- It is straightforward to check that this construction does not depend on the choice of primitive root $\omega$. 
Main Result

**Theorem**

If, for appropriate prime power $q$, $G$ is a Paley graph or Peisert graph, then for any $k, m \in \mathbb{N}$ and distinct $k$-vertex graphs $H_1, \ldots, H_m$ with $H_i \neq H_j$ for all distinct $i, j \in [m]$, if $\mathcal{B} := \bigcup_{i=1}^{m} (H_i(G) \cup \overline{H_i}(G))$ then $(V(G), \mathcal{B})$ forms a $2 - (|V(G)|, k, \lambda)$-design, where

$$\lambda = 2 \binom{k}{2} \binom{q}{2}^{-1} \sum_{i=1}^{m} |H_i(G)|.$$
An Example

(That’s too small to be interesting, but displays the idea well)

Take $G$ to be the 5-vertex Paley Graph:

Take $H$ to be the graph:
An Example

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So that $\overline{H}$ is:
**AN EXAMPLE**

(THAT’S TOO SMALL TO BE INTERESTING, BUT DISPLAYS THE IDEA WELL)

Take $G$ to be the 5-vertex Paley Graph:

![5-vertex graph](image)

The theorem says that $(V(G), H(G) \cup \overline{H}(G))$ forms:

2-(5, 3, 3)-design.
**AN EXAMPLE**

*(THAT’S TOO SMALL TO BE INTERESTING, BUT DISPLAYS THE IDEA WELL)*

Take $G$ to be the 5-vertex Paley Graph:

![Diagram of 5-vertex Paley Graph](image)

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![Diagram of a 5-vertex Paley Graph]

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![Diagram of a 5-vertex graph]

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![Graph Image]

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The theorem says that $(V(G), H(G) \cup \overline{H}(G))$ forms:

2-(5, 3, 3)-design.
A More Interesting Example

Using the 1148684789012489-vertex Paley graph,

2 - (1148684789012489, 3, 287171197253121) - DESIGN
Part 3: Sprint Relay Graphs
Part 3 is based on part of:

Non-noetherian groups and primality of their group algebras, with Tsunekazu Nishinaka, *Journal of Algebra*, Submitted on 12/29/15

arXiv:1602.03341
**Definition of a Sprint Relay Graph**

Let $G$ and $H$ be graphs with $V(G) = V(H) = V$, $E(G) = E$ and $E(H) = F$. If every component of $G$ is complete, and if $E \cap F = \emptyset$, then we call the graph with vertex set $V$ and edge set $E \cup F$ a *Sprint Relay Graph*, denoted $(G,H)$. 
**Definition of a Sprint Relay Graph**

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We view this as a simple 2-edge-colored graph with color classes $E$ and $F$. 
**Definition of a Sprint Relay Graph**

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We view this as a simple 2-edge-colored graph with color classes $E$ and $F$.

We call a color-alternating cycle in a Sprint Relay Graph a *Sprint Relay Cycle*.
Example

An example of a Sprint Relay Graph \((G, H)\) is given by the following. There, the edges of \(G\) are marked by solid segments and the edges of \(H\) are marked by dotted segments.
An example of a Sprint Relay Graph \((G, H)\) is given by the following. There, the edges of \(G\) are marked by solid segments and the edges of \(H\) are marked by dotted segments.

GREEN VERTICES INDUCE A SPRINT RELAY CYCLE.
Main Results

The first of our two main graph-theoretic results was:

**Theorem**

If \((G, H)\) is connected and each component of \(H\) is also complete, then \((G, H)\) has a Sprint Relay Cycle if and only if the sum of the number of components of \(G\) and the number of components of \(H\) is at most \(|V|\) (and \(|V| \geq 4\)).
Main Results

The first of our two main graph-theoretic results was:

**Theorem**

If \((G, H)\) is connected and each component of \(H\) is also complete, then \((G, H)\) has a Sprint Relay Cycle if and only if the sum of the number of components of \(G\) and the number of components of \(H\) is at most \(|V|\) (and \(|V| \geq 4\)).

We got a similar sufficiency result when \(H\) has complete multipartite components...
Part 4: Hamiltonian Cycles in Sparse Pseudorandom Bipartite Graphs
Part 4: Hamiltonian Cycles in Sparse Pseudorandom Bipartite Graphs

Part 4 is based on part of:

Sparse pseudorandom bipartite graphs are weakly pancyclic, with Felix Lazebnik and Andrew Thomason, In Progress

There Does Not Yet Exist a Preprint.
For $n$-vertex graph $G$, we let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ denote all (real) eigenvalues of $G$, ordered so that $\lambda_i(G) \geq \lambda_{i+1}(G)$ for all $i \in [n-1]$. 
Notation

• For $n$-vertex graph $G$, we let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ denote all (real) eigenvalues of $G$, ordered so that $\lambda_i(G) \geq \lambda_{i+1}(G)$ for all $i \in [n - 1]$.

• We call $G$ an $(n, d, \lambda)$-graph if $G$ is $n$-vertex, $d$-regular, and $\max\{|\lambda_i(G)| : i = 2, 3, \ldots, n\} = \lambda$. 
Notation

- For $n$-vertex graph $G$, we let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ denote all (real) eigenvalues of $G$, ordered so that $\lambda_i(G) \geq \lambda_{i+1}(G)$ for all $i \in [n-1]$.

- We call $G$ an $(n, d, \lambda)$-graph if $G$ is $n$-vertex, $d$-regular, and $\max\{|\lambda_i(G)| : i = 2, 3, \ldots, n\} = \lambda$.

- We call a graph $G$ a $\mathcal{B}(n, d, \lambda)$-graph if $G$ is an $n$-vertex graph which is bipartite, $d$-regular, and $|\lambda_3(G)| = \lambda$. 
Consider the following previous result:

\textbf{Theorem (Krivelevich and Sudakov*)}

If $G$ is an $(n, d, \lambda)$-graph, $n$ is sufficiently large, and

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$  \hspace{1cm} (1)

then $G$ is Hamiltonian.

\* M. Krivelevich and B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, \textit{Journal of Graph Theory} 42 (2003), no. 1, 17-33.
Consider the following previous result:

**Theorem (Krivelevich and Sudakov*)**

*If $G$ is an $(n, d, \lambda)$-graph, $n$ is sufficiently large, and

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$

then $G$ is Hamiltonian.*

**GIVES NO INFORMATION ABOUT BIPARTITE GRAPHS**
Ingredients of proof:

- The Expander Mixing Lemma*
- Pósa’s so-called rotation-extension technique**

** See, e.g., M. Krivelevich and B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, *Journal of Graph Theory* 42 (2003), no. 1, 17-33.
A bipartite analog of Krivelevich and Sudakov’s result:

Theorem

If $G$ is a $B(n, d, \lambda)$-graph, $n$ is sufficiently large, and

$$\lambda \leq \frac{(\log \log n)^2}{2000 \log n (\log \log \log n)} d,$$

then $G$ is Hamiltonian.
Ingredients of proof:

- We state and prove a bipartite analog of the Expander Mixing Lemma*
- We use a bipartite modification of Pósa’s so-called rotation-extension technique**
Thank you
Thank you!