UNIVERSITY OF DELAWARE
DEPARTMENT OF MATHEMATICAL SCIENCES
DISCRETE MATHEMATICS SEMINAR

Friday Apr. 25, 2003, 3:45pm, Room 436 Ewing Hall

Small regular graphs with girth 5, and their automorphism groups

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For given numbers k and g, the problem is to construct k-regular graphs with girth g and as few vertices as possible.

The smallest possible number of vertices in such a graph is denoted by $f(k, g)$. The Moore bound states that $f(k, g) \geq n(k, g)$, where

$$n(k,g) = \begin{cases} \dfrac{k(k-1)^{\frac{g-1}{2}} - 2}{k-2} & \text{if } g \text{ is odd} \\ \dfrac{2(k-1)^{\frac{g}{2}} - 2}{k-2} & \text{if } g \text{ is even.} \end{cases}$$

A graph for which equality holds is called a Moore graph. Hoffman and Singleton (1960) proved that a Moore graph with girth 5 has degree either $k = 2$ (5-cycle), $k = 3$ (Petersen graph), $k = 7$ (Hoffman-Singleton graph), or possibly $k = 57$. The existence of a Moore graph of degree 57 remains an intriguing open problem. It has been shown that if it exists it cannot have a large automorphism group (in contrast to the Petersen graph and the Hoffman-Singleton graph).

The value of $f(k, 5)$ is known for all $k \leq 7$. But $f(k, 5)$ is not known for any $k \geq 8$. Some best known upper bounds are found by recent constructions:

- G. Royle has shown that $f(8, 5) \leq 80$ by constructing an 8-regular Cayley with girth 5 and order 80.
- G. Exoo constructed graphs with two vertex orbits and thus found upper bounds on $f(k, 5)$ for $k = 10, 11, 12, 13, 14$.

We present a new construction of regular graphs with girth 5. The graphs constructed includes the Petersen graph, the Hoffman-Singleton graph, Royle's graph and a variation of the graphs constructed by Exoo. The construction is based on Cayley graphs and relative difference sets with $\lambda = 1$. (A relative difference set in an (abelian) group G relative to a subgroup H is a set $S \subseteq G$ such that elements in H can not be written as $x - y$, where $x, y \in S$ and elements in $G \setminus H$ can be written as $x - y$ for exactly λ pairs (x, y) in S.)