ZDM Mathematics Education (2009) 41:777–792
DOI 10.1007/s11858-009-0197-1

ORIGINAL ARTICLE

How a standards-based mathematics curriculum differs from a traditional curriculum: with a focus on intended treatments of the ideas of variable

Bikai Nie · Jinfa Cai · John C. Moyer

Accepted: 22 June 2009 / Published online: 9 July 2009

Abstract Analyzing the important features of different curricula is critical to understand their effects on students' learning of algebra. Since the concept of variable is fundamental in algebra, this article compares the intended treatments of variable in an NSF-funded standards-based middle school curriculum (CMP) and a more traditionally based curriculum (Glencoe Mathematics). We found that CMP introduces variables as quantities that change or vary, and then it uses them to represent relationships. Glencoe Mathematics, on the other hand, treats variables predominantly as placeholders or unknowns, and then it uses them primarily to represent unknowns in equations. We found strong connections among variables, equation solving, and linear functions in CMP. Glencoe Mathematics, in contrast, emphasizes less on the connections between variables and functions or between algebraic equations and functions, but it does have a strong emphasis on the relation between variables and equation solving.

1 Introduction

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published the first round of its *Standards* documents (e.g., NCTM, 1989, 1991, 1995), which provided recommendations for reforming and improving K-12 school mathematics. With extensive support from the National Science Foundation (NSF), a number of school mathematics curricula were developed and implemented to align with the recommendations of the *Standards*. In the past decade, there have been heated discussions about the benefits of using these so-called *Standards*-based mathematics curricula in the USA (e.g., Herman et al., 2006; Schoenfeld, 2006; Senk & Thompson, 2003; Wu, 1997). The *Standards*-based, NSF-funded curricula claim to have different learning goals from traditional mathematics curricula, and the layout and organization of reform texts appear quite different from traditional texts. As a result, some parents, professionals and school communities who are comfortable with the goals and organization of traditional curricula challenge both the goals and the efficacy of these new curricula (Cai, 2003).

We should first clarify the two terms in this paper: traditional curriculum and *Standards*-based curriculum. On the one hand, NSF-funded curricula, which commonly are referred to as *Standards*-based, build students' understanding of important mathematics through explorations of real-world situations and problems. On the other hand, most publishers of so-called traditional curricula also claim that their curricular materials are *Standards*-based. In fact, their teachers' guides usually describe specifically how each unit in each chapter of the textbooks corresponds to the NCTM content and process standards (e.g., Bailey et al., 2006a, b, c). Nonetheless, unlike the NSF-funded curricula, these curricula are commonly referred to as "traditional" rather than "*Standards*-based." In keeping with this common practice, we refer to the NSF-funded and the other curricula discussed in this paper as "*Standards*-based" and "traditional," respectively.

How does a *Standards*-based curriculum really differ from a traditional curriculum? What are the important features of the NSF-funded *Standards*-based curricula that

B. Nie · J. Cai (✉)
University of Delaware, Newark, DE, USA
e-mail: jcai@math.udel.edu

J. C. Moyer
Marquette University, Milwaukee, WI, USA

distinguish them from so-called traditional curricula? Answering these research questions requires fine-grained analyses of the important features of both types of curricula. So far, only a few studies have been done to identify those important features of *Standards*-based curricula that distinguish them from traditional curricula, and fewer studies focus their research on the algebraic strand within those curricula. The purpose of this paper is to present a fine-grained analysis that compares the treatment of the concept of variable in the NSF-funded Connected Mathematics Program (CMP) (e.g., Lappan et al., 2002a) with that of the more traditionally based Glencoe *Mathematics: Concepts and Applications* (Bailey et al., 2006a, b, c) curriculum.[1]

The CMP curriculum is a complete NSF-funded middle school mathematics program that is commonly referred to as *Standards*-based. Like other curricula that are widely regarded to be *Standards*-based, the CMP curriculum is designed to build students' understanding of important mathematics through explorations of real-world situations and problems. Students using the CMP curriculum are guided to investigate important mathematical ideas and ways of thinking as they try to understand and make sense of real-world situations.

The Glencoe curriculum is also a complete middle school mathematics program. Like CMP, the Glencoe authors claim that student learning with the curriculum is "relevant real-life, problem-based." Furthermore, the Glencoe curriculum claims to be "…designed to utilize several important research-based strategies to reinforce the high goals set by…the *Principles and Standards for School Mathematics*" (Bailey et al., 2006a, b, c, p. T20). The teacher's handbook even provides a table showing NCTM standards addressed by each of the lessons. Nonetheless, the Glencoe curriculum is considered by mathematics educators to be traditional, rather than *Standards*-based.

[1] The research reported here is part of a large project designed to longitudinally compare the effects of the Connected Mathematics Program (CMP) with the effects of more traditional middle school curricula on students' learning of algebra (Cai & Moyer, 2006). In the large project (Longitudinal Investigation of the Effect of Curriculum on Algebra Learning, LieCal Project), we investigate not only the ways and circumstances under which the CMP and other curricula like Glencoe Mathematics can or cannot enhance student learning in algebra, but also the characteristics of the curricula that lead to student achievement gains. In 2006 and 2009, the authors published revised editions of the CMP curriculum under the name CMP2. This article is based on the CMP curriculum because the students in the LieCal project used CMP, and not CMP2. LieCal Project is supported by a grant from the National Science Foundation (ESI-0454739). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

2 Theoretical considerations of the study

2.1 Analyzing the curriculum

Research has consistently shown that student achievement is highly correlated with the intended curricular treatment (Schmidt et al., 2002). Findings of cross-national studies suggest that differences in mathematical performance across countries are due, in part, to differences in the curricula of the different countries (e.g., Cai, 1995; Ginsburg & Clift, 1990; Jackson, 1992).

Researchers analyzed the intended curricular treatment mainly using content analysis, although some adopt a socio-linguistic approach to discuss socio-cultural influences of curricular treatment. For example, Dowling (1996) developed a sociological framework to analyze how school mathematics texts produce and reproduce social structure by subtly regulating "…who can say or do or believe what," (p. 408). In particular, he analyzed the case of two books in the School Mathematics Project (SMP), which he claims constructs a hierarchy of readers that helps maintain the cultural division between white collar workers and manual laborers. Herbel-Eisenmann (2007) used a discourse analysis approach and found that the use of traditional forms of discourse in *Standards*-based curriculum materials leads to a mismatch between the effects of the text on learners and the goal put forth by NCTM (1991) to shift the authority away from the teacher and the textbook and toward student mathematical reasoning and justification.

Given that the focus of this paper, we review studies using content analysis. Using content analysis, researchers can consider various factors, including mathematical accuracy, characteristics of mathematical problems and selection of mathematical topics (Cai, Lo, & Watanabe, 2002; Li, 2000; NRC, 2004). For example, Li (2000) analyzed the mathematics problems in selected US and Chinese textbooks. He found a series of significant differences between the US and the Chinese textbooks with respect to the problems' performance requirements.

Cai et al. (2002) examined the intended treatment of arithmetic average in two US *Standards*-based curricula (one of which is the CMP curriculum), a more traditional US curriculum and three Asian curricula. They found that although the goals for the learning of average in these curricula were similar, the instructional treatments were quite different. In fact, the more traditional curriculum of the US focuses more on understanding the concept of average as a computational algorithm than on understanding the concept of average as a representative of a data set; however, the two US *Standards*-based curricula focus more on the latter exposition of the concept. The study, though small in scale, not only provides insightful information on the features of arithmetic average in *Standards*-based curricula, but more importantly, it establishes the feasibility of identifying important features of *Standards*-based curricula that distinguish them from other curricula.

Kulm, Morris, and Grier (1999) analyzed the algebra strands of more than 10 US middle school curricula in terms of two algebraic ideas: algebra graph concepts and algebra equation concepts. They employ content coverage and instructional quality as two dimensions to examine how well the textbook addresses the content coverage related to algebra graph concepts and algebra equation concepts. The three levels were used to describe the degree of content coverage: (A) most content, (B) partial content and (C) minimal content. Four levels were used to describe the instructional quality: (A) high potential for learning to take place, (B) some potential for learning to take place, (C) little potential for learning to take place and (D) not present. Two of the curricula compared are the CMP curriculum and the curriculum called Mathematics: Applications and Connections (MAC curriculum) (Collins et al., 1998a, b, c). Kulm et al. (1999) found that CMP curriculum not only has more content coverage of algebra concepts related to graphs and equations than the MAC curriculum, but also showed much higher potential for students' learning than the MAC curriculum.

Haggarty and Pepin (2002) analyzed the intended treatment of the concept of "angle" by focusing on the mathematics that is made available in English, French and German textbooks. They explored the following aspects of the mathematical exposition provided in the texts: the technical vocabulary that is used, the mathematical notations that are utilized, the completeness of the explanations preceding the exercises and the connections made with other topics. They found that the textbooks in different countries not only provide students with different mathematics, but they also offer students different opportunities to learn. For example, the French textbook uses a relatively complex mathematical notation for angles, but the English textbook does not use any notation to name angles.

Cai (2004a) and Cai et al. (2005) developed an analytic framework for analyzing curriculum, which concentrates on the goal specification, content coverage and process coverage of the curricula being analyzed. Using this framework, they conducted a series of studies with a focus on algebraic thinking (e.g., Cai, 2004a, b; Cai et al., 2005; Moyer, Huinker, & Cai, 2004; Ng, 2004) that yielded a number of meaningful findings: (1) the US elementary mathematics curriculum *Investigations* accomplishes the teaching of "change" from informal models based on children's intuitions, but it does not progress to symbolic or formal work; (2) the Russian curriculum of Davydov engages children in a focused analysis and description of the quantitative world as the starting point for developing algebraic thinking, and symbolic algebra is introduced in the first grade; (3) in the Chinese elementary mathematics curriculum, children's function sense is developed by variable and function ideas that permeate the curriculum, but variable and function ideas are not formally defined.

In this study, we focus on the intended treatment of the ideas of variable to examine how CMP and Glencoe Mathematics differ from each other. We focus on variable because of its importance in mathematics and in the learning of algebra. Meanwhile, our analysis is guided by a framework developed to examine how algebraic ideas are introduced and developed in various curricula (Cai, 2004a; Cai et al., 2005).

2.2 Focusing on the ideas of variable

Variable is one of the most important algebraic ideas (NCTM, 2000; Schoenfeld & Arcavi, 1988). A major difference between arithmetic and algebra is the involvement of variables in algebra. In algebra, the concept of variable can be understood in different ways. For example, variables can be introduced in a curriculum as quantities whose values may change or vary according to circumstances. From this perspective, variables can represent many numbers simultaneously; they have no place value and can be selected arbitrarily. In the mathematics education community, we have not had any consistent conceptions for the following pairs of concepts: letters and variables, unknowns and variables, placeholders and variables (Janvier, 1996). Some educators and curriculum developers believe that both words in each of the pairs mean the same thing, but others believe that they represent different concepts. For example, Wheeler (1996) thinks the letter "x" can stand for an as yet unknown number, a general number or a variable. Obviously, from Wheeler's point of view, a letter and a variable are not the same thing, and a variable is different from an unknown and from a general number. As part of a larger study, when Schoenfeld and Arcavi (1988) asked a diverse group of people (mathematicians, mathematics educators, computer scientists, linguists, logicians, and so forth) to describe the concept of variable in one word, they produced the following list: symbol, placeholder, pronoun, parameter, argument, pointer, name, identifier, empty space, void, reference and instance. Interestingly, none of the subjects used the word *unknown* to describe the concept of variable. Schoenfeld and Arcavi (1988) explained that the omission occurred because the word *unknown*, which connotes something that has a fixed value that one does not yet know, did not match the subjects' conception of variable as something that varies or has multiple values. Schoenfeld and Arcavi also listed 10 different meanings of variable from a variety of sources, making the point that mathematicians use the term variable differently in different contexts, and that this practice makes it difficult for mathematics educators to define the word variable, and even more difficult for students to learn the concept.

We have alluded to the fact that there is no consensus in the mathematics education community on a single definition of variable that should be used in algebra textbooks. One reason for this is that there is similarly no consensus about the role that the study of algebra should play in precollege mathematics. Different conceptions of algebra are better served by some interpretations of variable than others (Usiskin, 1988). For example, one conception of algebra that is favored by many educators is that algebra is generalized arithmetic. For this conception, the interpretation of a variable as a pattern generalizer is preferable to other interpretations. A different conception of algebra is that it is the study of procedures for solving certain kinds of problems. In this case, the most useful meaning of variable is that of an unknown or constant. The concept of algebra as the study of relationships among quantities, on the other hand, is best served by emphasizing the use of variables as arguments or parameters. Finally, the conception of algebra as strictly the study of structures is best served if variables act as arbitrary symbols or marks on a paper.

Since there is no agreement among mathematics educators regarding the definition of the term variable, it is important to determine which, if any, of the commonly used interpretations of variable align with the goals of the two curricula being analyzed. Variables have been used in middle school mathematics curriculum materials in each of the following three ways:

1. Variables viewed as pattern generalizers (e.g., in generalizing $3 + 5 = 5 + 3$ to the pattern $a + b = b + a$) or as representatives of ranges of values (e.g., in using $3t + 6$ to represent the possible values that can result when 6 is added to 3 times a quantity);
2. Variables viewed as placeholders or unknowns in naked equations (e.g., as in $x + 6 = 21$), or in equations translated from a word problem (e.g., "In how many years will your 6-year-old sister be 21?");
3. Variables used to represent relationships, such as in the use of $y = 9x-43$ to represent an equation of the line with slope 9 that goes through the point (5, 2), or as in the use of $C = 15\ N$ to represent the relation between the number of \$15 tickets (N) and their total cost (C).

In this paper, we examine which of these meanings of variable align with the goals and approaches of the portions of the CMP and Glencoe Mathematics curricula that introduce the concept of variable, and that develop concepts related to algebraic equations and linear functions.

3 Methods

3.1 The curriculum materials

The Connected Mathematics Program (CMP) was funded by the National Science Foundation between 1991 and 1996, and revised between 2000 and 2006, to develop a complete middle school mathematics curriculum. The result was the *Connected Mathematics* textbook series. CMP is problem-centered, and it emphasizes students' inquiries, discoveries and understanding of mathematical ideas through the investigation of rich problem situations (Lappan et al., 2002h). As each algebraic unit progresses and the number of problem situations that students have been exposed to increases, the intent is that students will learn algebraic concepts and procedures simultaneously, making meaningful connections among them. In addition, students are expected to discern informal justifications of basic algebraic properties, procedures and solutions to problems.

The CMP mathematics textbooks[2] are divided into a series of independent units, with eight units for each of the three middle school grade levels (sixth, seventh and eighth grades). Each unit is published in a stand-alone booklet that helps develop one of the four mathematical strands in CMP: number and operation (six units), geometry and measurement (six units), data analysis and probability (five units) and algebra (seven units). The algebra units are *Variables and Patterns* (grade 7: Introducing Algebra), *Moving Straight Ahead* (grade 7: Linear Relationships), *Thinking with Mathematical Models* (grade 8: Linear and Inverse Variation), *Looking for Pythagoras* (grade 8: The Pythagorean Theorem),[3] *Growing, Growing, Growing* (grade 8: Exponential Relationships), *Frogs, Fleas, and Painted Cubes* (grade 8: Quadratic Relationships), and *Say it with Symbols* (grade 8: Making Sense of Symbols). Most of the algebraic content in CMP is developed in the eighth grade. However, in this article we focus our attention on the two-seventh grade algebra units (*Variables and Patterns* and *Moving Straight Ahead*), because our main interest lies in how the concept of variable is formally introduced in CMP. Nonetheless, we analyzed all seven algebra units for this paper (Lappan et al., 2002a, b, c, d, e, f, g).

[2] As we indicated before, this article is based on the CMP curriculum, because the students in the LieCal Project used CMP, and not CMP2. Furthermore, we can accomplish our purpose, which is to examine how a representative NSF-funded *Standards*-based middle school curriculum differs from a representative traditional curriculum, using either CMP or CMP2 as the *Standards*-based curriculum.

[3] In CMP2, this unit is categorized as a geometry unit. Nonetheless, like the CMP curriculum, the CMP2 curriculum has seven algebra units because the CMP2 authors added a new eighth grade algebra unit (See Lappan et al., 2006).

The Glencoe Mathematics curriculum is widely used in middle school mathematics classrooms. The curriculum includes three separate textbooks, called *Glencoe: Mathematics Applications and Concepts: Course 1*, *Course 2* and *Course 3*. The three courses correspond to grades six, seven and eight, respectively. Each course comprises five or six units. For example, Course 1 consists of six units: (1) "Whole numbers, Algebra, and Statistics"; (2) "Decimals"; (3) "Fractions"; (4) "Algebra"; (5) "Ratio and Proportion"; and (6) "Measurement and Geometry" (Bailey et al., 2006a). The units are composed of two or three chapters, each of which is divided into short lessons that present bite-sized, incremental portions of concepts and procedures for students to digest.

The Glencoe Mathematics textbooks include formal definitions of mathematical concepts and worked-out examples as the backbone of the curriculum. It appears that the textbooks are organized in this way because the authors expect students to learn concepts and procedures by studying definitions and by examining and imitating solutions provided in the example problems. Completely worked-out examples with clear explanations are provided in the textbooks and are followed by student exercises, most of which are closely tied to the examples (Bailey et al., 2006a, b, c). In fact, the "Practice and Applications" problems at the end of each lesson come with a "Homework Help" chart that keys most of the problems to their corresponding worked-out examples in the lesson.

We decided which lessons to analyze based not only on the titles of the lessons, but also on the lesson objectives that are specified in the *Chapter Overview* sections of the teacher's wraparound edition of the textbook. For example, we chose to analyze two of the eight lessons in Chapter 1 (*Number Patterns and Algebra*) of Unit 1 (*Whole Numbers, Algebra, and Statistics*) of Course 1 (Grade 6). They are: Lesson 1–6 (*Algebra: Variables and Expression*), whose stated objective is to "Evaluate algebraic expressions," and Lesson 1–7, (*Algebra: Solving Equations*), whose stated objective is to "Solve equations by using mental math and the guess and check strategy." None of the other lessons in Chapter 1 has titles or stated objectives that are explicitly algebraic.

3.2 Analysis plan

Our analysis was guided by a framework developed to examine how algebraic ideas are introduced and developed in various curricula (Cai, 2004a; Cai et al., 2005). The framework concentrates on the goal specification, content coverage and process coverage of the curricula being analyzed. In this paper, the focus of our analysis is to convey how the concept of variable is treated in the CMP and Glencoe Mathematics curricula. To do so, we first identify the learning goals of the two curricula that are related to the concept of variable. In particular, we analyzed the learning goals using the three previously described conceptions of variables: (1) variables viewed as pattern generalizers or representatives of ranges of values, (2) variables viewed as placeholders or unknowns, (3) variables used to represent relationships. Then we examined when and how the concept of variable is introduced in the two curricula. Lastly, we examined how the concept of variable is related to algebraic equations and linear functions. In this examination, our focus is on how algebraic equation and linear functions are introduced building on the concept of variable.

In this article, our analysis focuses on the following research questions:

1. What are the differences between the learning goals for the concept of variable in the two curricula?
2. How is the concept of variable introduced in the two curricula?
3. How is the concept of variable related to the development of equation solving in the two curricula?
4. How is the concept of variable related to the development of linear functions in the two curricula?

4 Results

4.1 Learning goals for the concept of variable

CMP learning goals can be found in the implementation guide (Lappan et al., 2002h), in the lesson planner (Lappan et al., 2002i), in the Teacher's Guide to each unit and in the student texts themselves. The learning goals for the Glencoe Mathematics curriculum are given in the teacher's wraparound edition of each course. Table 1 shows the focus of the learning goals related to the concept of variable in the CMP and Glencoe curricula.

The learning goals related to the acquisition and use of the concept of variable in the CMP curriculum focus on the use of variables to represent relationships. The following learning goals are representative of those related to the

Table 1 Focus of the learning goals related to the concept of variable in the CMP and Glencoe Mathematics curricula

Conceptions of variables	CMP	Glencoe
The learning goals characterize variables as pattern generalizers or as being used to represent ranges of values	✔	✔
The learning goals characterize variables as placeholders or unknowns		✔
The learning goals characterize variables as being used to represent relationships	✔	

concept of variable in CMP: "[to] search for patterns of change that **show relationships** among the variables," (Lappan et al., 2002i, p. 71), "[t]o understand that variable is a quantity that changes and to recognize the variables in the real world," (Lappan et al., 2002i, p. 73), and "[t]o identify variables and determine an appropriate range of values for independent and dependent variables," (Lappan et al., 2002i, p. 103). These learning goals are very explicit in their expectation that students understand that variables are used to represent relationships. We could not find any goal statements in CMP that suggest that variables should be viewed as placeholders or unknowns.

In contrast, almost all the learning goals about the concept of variable in Glencoe Mathematics describe a variable as a placeholder or an unknown. For example, the *Mathematical Content and Teaching Strategies* section of the teacher's wraparound edition says the following about Lesson 1–6 of Course 1: "… students first explore the use of models to stand for unknown quantities. They then transfer this concrete sense to an understanding of the function of variables in algebraic expressions," (Bailey et al., 2006a, p. 4D). Furthermore, the Glencoe learning goals that involve the concept of variable are typically written in an equation-solving context. For example, the following statement refers to Lesson 1–7 of Course 1: "In earlier grades, students were exposed to the concept of missing parts of equations as represented by boxes and circles. They learn how variables serve the same function in an algebraic equation," (Bailey et al., 2006a, p. 4D). Here is a second example, taken from the grade 8 teacher wraparound edition: "A problem like () + 6 = 8 that they might have included in an earlier course is now written with a variable as $x + 6 = 8$," (Bailey et al., 2006c, p. 4C). It is apparent from these examples that an important learning goal in Glencoe Mathematics is for students to understand that variables are placeholders or unknowns.

To complement CMP's emphasis on the use of variables to represent relationships, CMP encourages students to view variables as pattern generalizers or representatives of ranges of values. This is done by requiring students to "…search for patterns of change that show relationships among the variables," (Lappan et al., 2002i, p. 71).

Glencoe Mathematics does not have specific learning goals that suggest that variables should be viewed as pattern generalizers or representatives of ranges of values. However, we can find instances in the Glencoe Mathematics curriculum where variables are used as pattern generalizers. For example, in sixth grade a proportion is defined as "…an equation stating that two ratios are equivalent," and it is accompanied by the following use of variables as pattern generalizers:

"$a/b = c/d$, $b \neq 0$, $d \neq 0$," (Bailey et al., 2006a, p. 386).

4.2 Introduction of the concept of variable

4.2.1 Variable in CMP

The CMP formally introduces the concept of variable in grade seven, while Glencoe Mathematics introduces the concept of variable in grade six. Although both curricula formally define the term *variable*, the definitions of variable provided by CMP and Glencoe Mathematics are very different. CMP defines the term *variable* in conjunction with its connection to coordinate graphs: "A **variable** is a quantity that changes or *varies*. … A **coordinate graph** is a way to show the relationship between two variables," (Lappan et al., 2002a, p. 7). Glencoe Mathematics defines a variable as "…a symbol, usually a letter, used to represent a number," (Bailey et al., 2006a, p. 28). From these definitions, it is clear that CMP uses variables to represent relationships, while variables are viewed as placeholders or unknowns in Glencoe Mathematics.

The CMP curriculum's definition of variable as a quantity rather than a symbol makes it convenient to use variables informally in relationships long before it introduces the concept of variable formally in the seventh grade. In Investigation 4 ("Coordinate Graphs") of the sixth grade unit *Data About Us* (Lappan et al., 2002j), students analyze data by constructing coordinate graphs to explore relationships among quantities listed in tables (e.g., distance and time from school, height and foot length). This is done by labeling the horizontal and vertical axes with the names of the quantities, plotting data points, and observing that there is a relationship between the quantities. Sometimes the relationship is qualitative ("Students who live further away from school generally spend more time getting to school" (p. 45)); sometimes the relationship is quantitative ("Height is generally about 6 to 6-1/2 times foot length" (p. 44)).

A year later, when CMP formally introduces the concept of variable in the seventh grade unit *Variables and Patterns* (Lappan et al., 2002a), only the word *variable* is new. This is because the formal use of variables in the first three investigations of the seventh grade unit ("Variables and Coordinate Graphs," "Graphing Change," and "Analyzing Graphs and Tables") is the same as the informal use of quantities in the sixth grade. That is, students use quantity names (now called "word names for the variables") as before, viz, to describe relationships in words and to label columns of data tables and axes of coordinate graphs.

It is not until Investigation 4 ("Patterns and Rules") of *Moving Straight Ahead* that the students are finally introduced to the use of symbols for variables (Lappan et al., 2002b). Investigation 4 provides the following rationale for using symbols: "A shorter way to write rules using variables is to replace the word names for the variables with single letters," (p. 50). One of the application problems, which appears at the end of the investigation that introduces the concept of variable is shown in Fig. 1. Question *a* in the CMP part of Fig. 1 illustrates the curriculum's emphasis on understanding the concept of variable by using real-world situations. Question *b* illustrates how CMP students can experience the variability and interrelationship of variables using a real-world context to develop scatterplots.

In CMP, the development of the concept of variable underscores the changing or varying nature of variables and emphasizes that expressing relationships between variables is at the heart of algebra. Rather than developing the concept of variable by introducing algebraic equations immediately, CMP introduces the terms *independent variable* and *dependent variable*. Then relationships between independent and dependent variables are emphasized with graphs and tables of real-world quantities.

4.2.2 Variable in Glencoe Mathematics

Glencoe Mathematics defines a variable as a symbol (or letter) used to represent a number, and the examples that follow show students how to evaluate algebraic expressions for given values of the variables (see Fig. 1). These examples give the impression that variables and numbers can be interchanged. This is because every variable (letter) is assigned only one number. Letters used in this way in equations are often called *unknowns* and are not universally considered to be variables, because they are thought of by many as having *fixed* values that we do not yet know (Usiskin, 1988; Schoenfeld & Arcavi, 1988).

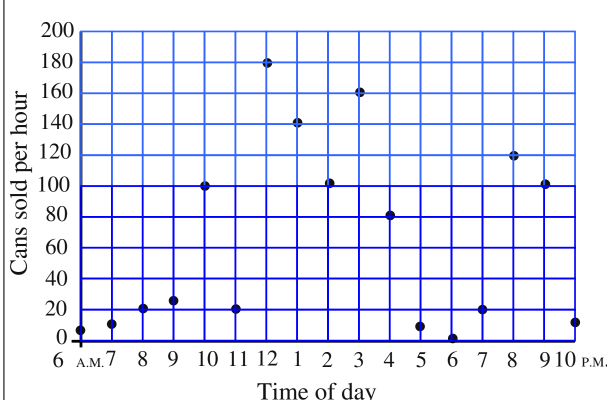
Based on an analysis of the problems used in the introduction of variable in their curricular materials, we believe that it is probable that the initial conceptualizations of variable for CMP students and Glencoe Mathematics students will be different. CMP provides an opportunity for students to understand variables from a dynamic perspective by analyzing the relationships between them. However, Glencoe Mathematics imparts a static perspective to the concept of variable by giving the impression that every variable is a letter that has a fixed value.

4.3 The idea of variable in equation solving

4.3.1 Equation solving in the CMP curriculum

The CMP curriculum encourages students to solve equations by building on thought processes similar to those of the "algebraic group" distinguished by Kieran (1988). That

Fig. 1 Sample problems to introduce the concept of variable in CMP and Glencoe Mathematics curricula (Lappan et al., 2002a, p. 11; Bailey et al., 2006a, p. 29)

CMP	**Glencoe Mathematics**
The graph below shows the numbers of cans of soft drink purchased each hour from a school's vending machine in one day (6 means the time from 5:00 to 6:00, 7 represents the time from 6:00 to 7:00, and so on).  *a. The graph shows the relationship between two variables. What are the variables?* *b. Describe how the number of cans sold changed during the day. Give an explanation for why these changes might have occurred.*	*Evaluate algebraic expressions* *(1)Evaluate* $16+b$ *if* $b=25$; *(2)Evaluate* x-y *if* $x=64$ *and* $y=27$.

is, CMP attempts to develop students' understanding of the role of inverse operations in equation solving at the same time that it develops their facility to carry them out. To do so, CMP continues its emphasis on interpreting mathematics through the lens of real-world applications by explaining the use and meaning of inverse operations in a contextual way.

Strictly speaking, CMP does not provide a definition for the term *equation*. However, it introduces the concept of equation in the following way: "Sometimes the relationship between two variables can be described with a simple rule. …Rules, like those above, that are expressed with mathematical symbols are sometimes referred to as equations or formulas," (Lappan et al., 2002a, p. 49). The idea that equations are rules that describe relationships between two variables is of utmost importance in CMP. To establish this notion in students' minds, equations as descriptors of relationships between variables are formally introduced and developed in the first unit of the seventh grade (*Variables and Patterns*), but linear equation *solving* is deferred until the second semester of the seventh grade in the unit *Moving Straight Ahead* (Lappan et al., 2002b). Even then, symbolic equation solving is not studied until halfway through the unit. The first three investigations in *Moving Straight Ahead* use real-world situations to explore the concept of linearity, including graphical and tabular methods for solving linear equations in one variable. When the symbolic solution of linear equations is finally introduced in *Moving Straight Ahead*, the symbolic manipulations themselves are not used to justify the equation-solving steps, as they are in the Glencoe Mathematics curriculum. Instead, the CMP curriculum justifies the equation-solving manipulations through contextual sense-making of the symbolic method. In other words, CMP uses real-life contexts to help students understand the meaning of each step of the symbolic method of equation solving, including why inverse operations are used (See Fig. 2).

Although the variable N in Fig. 2 is an unknown, the CMP curriculum downplays the view of N as a placeholder or unknown. We have already mentioned that the symbolic method for solving linear equations shown in Fig. 2 is preceded in the CMP curriculum by an investigation of methods for solving linear equations in one variable, using the table or graph of the corresponding equation in two variables (Lappan et al., 2002b, p. 54). It is also worth mentioning that not only are there practice problems in CMP, such as "Find x, if $326 = 4x$," which require students to solve equations in one variable, but also there are practice problems like "… [D]o parts a and b by using the symbolic method and by using a graphing calculator. …$y = x - 15$, (a) Find y if $x = 9.4$; (b) Find x if $y = 29$," (p. 59). These require students to use equations that describe the relationship between two variables. Obviously, the second type of problems is meant to prompt students to recognize the variability of the values that a variable represents. In so doing, these problems stress a view of variable that is consonant with the view of a variable as an unknown, but that is more inclusive, namely, that when

Fig. 2 An example of equation solving in CMP (Lappan et al., 2002b, p. 55)

The Unlimited Store allows any customer who buys merchandise costing over \$30 to pay on the installment plan. The customer pays \$30 down and then pays \$15 a month until the item is paid for. Suppose you buy a \$195 CD-ROM drive from the Unlimited Store on an installment plan, How many months will it take you to pay for the drive? Describe how you found your answer.

Thinking	***Manipulating the Symbol***
"I want to buy a CD-ROM drive that costs \$195. To pay for the drive on the installment plan, I must pay \$30 down and \$15 a month."	$195 = 30 + 15N$
"After I pay the \$30 down payment, I can subtract this from the cost. To keep the sides of the equation equal, I must subtract 30 from both sides."	$195 - 30 = 30 - 30 + 15N$
"I now owe \$165 which I will pay in monthly installments of \$15."	$165 = 15N$
"I need to separate \$165 into payments of \$15. This means I need to divide it by 15. To keep the sides of the equation equal, I must divide both sides by 15."	$\frac{165}{15} = \frac{15N}{15}$
"There are 11 groups of \$15 in \$165, so it will take 11 months."	$11 = N$

variables are used to represent relationships, they are quantities that vary.

4.3.2 Equation solving in Glencoe Mathematics

The definitions of equation ("…a sentence that contains an **equals** sign") and solve ("When you replace a variable with a value that results in a true sentence…) are given in Glencoe Lesson 1–7 ("Algebra: Solving Equations"). Unlike CMP, the Glencoe curriculum does not encourage students to solve equations by using thought processes similar to the "algebraic group," as distinguished by Kieran (1988). Instead, Glencoe Mathematics first asks students to solve equations mentally (see Fig. 3), which is typical of the thinking in Kieran's "arithmetic group." This strategy is aligned with Glencoe's stated goal to teach students that variables serve the same function in an algebraic equation that boxes and circles did in arithmetic, namely to represent the missing parts of equations.

The example in Fig. 3 is taken from Lesson 1–7, which is the lesson immediately following the students' introduction to the concept of variable in the Glencoe Mathematics. Placing equation solving at the heels of the introduction of variables is consistent with Glencoe's view of variable as a placeholder or unknown. Developing students' equation-solving ability by using mental math shows that Glencoe views algebra as generalized arithmetic. However, reflection on Kieran's research (above) advises us that using mental math to solve equations builds on students' knowledge of the computational aspects of arithmetic rather than on its structure. Furthermore, while it is true that if students "THINK 12 equals 5 plus what number*?"* they will find the letter *h* equals 7 based on their arithmetic experience. However, most students will find the solution so quickly that the letter *h* is not a true unknown for them. At most, it is a placeholder. Those who do not immediately know the solution do not need to use the inverse operation to find the solution. The Glencoe curriculum provides an alternative strategy for them, namely, "Use Guess and Check" (p. 35). Using this method, they might process the entire series of additions: $5 + 1 = 6$; $5 + 2 = 7$; $5 + 3 = 8$ … $5 + 7 = 12$ until they arrive at the correct answer. However, it is unlikely that students using this method would connect the variable *h* with any number except for the number 7, even if they considered the entire collection of numbers 1, 2, 3,…, 7. Thus, it is unlikely that students using either the mental math or guess and check methods would build a conception of variable that incorporates the notion of variation.

Solve $12 = 5 + h$ mentally.	
$12 = 5 + h$	THINK 12 equals 5 plus what number?
$12 = 5 + 7$	You know that $12 = 5 + 7$.
$h = 7$	The solution is 7.

Fig. 3 An example of equation solving in Glencoe Mathematics (Bailey et al., 2006a, p. 35)

Glencoe Mathematics formally introduces equation solving with inverse operations by way of an activity that uses a cup to stand for an unknown (see Fig. 4). The cups and counters used as manipulatives in the activity are simply exact representations of the symbols. The manipulatives only illustrate each step of the symbolic manipulations rather than clarify their meaning. For example, although the introduction to the activity states that you need to keep the equation balanced, no rationale is illustrated or provided in the activity itself for "*Remove 5 counters from each side to get the cup by itself.*" In fact, removing five counters from only the left side would also "*…get the cup by itself.*" Kieran's (1988) research implies that members of both the arithmetic and algebraic groups benefit from persistent emphasis on the rationale for performing the same operation on both sides, namely to keep the equation in balance and to maintain the solution unchanged throughout the equation-solving process.

Glencoe Mathematics uses diagrams specifically to show the procedure for translating words into two-step equations (see Fig. 5). There are two interesting points about this procedure, which is taken from Sect. 10-3 of Glencoe's eighth grade curriculum. First is that the variable "*n*" is treated as an unknown, which illustrates that the interpretation of variable as unknown or placeholder remains consistent across the grade levels of the Glencoe curriculum. Second is that the Glencoe curriculum stresses the difference between translating two-step and one-step equations. This kind of distinction is not made in the CMP curriculum.

In general, Glencoe Mathematics focuses on the "operational aspect" of equation solving. Overall, Glencoe Mathematics very specifically categorizes equation-solving lessons in the textbook by the type of operations involved in solving the equations, as well as by the number of steps involved in the equation-solving and equation writing processes. However, it rarely connects equation solving with the ideas of function or other algebraic ideas. This finding confirms that Glencoe Mathematics treats variables as unknowns when equations are introduced and continues to do so throughout the curriculum, without emphasizing the interpretation of variable as a quantity that varies.

4.4 The idea of variable in linear functions

4.4.1 Linear functions in CMP

CMP employs multiple representations to informally introduce the concept of function and to help students

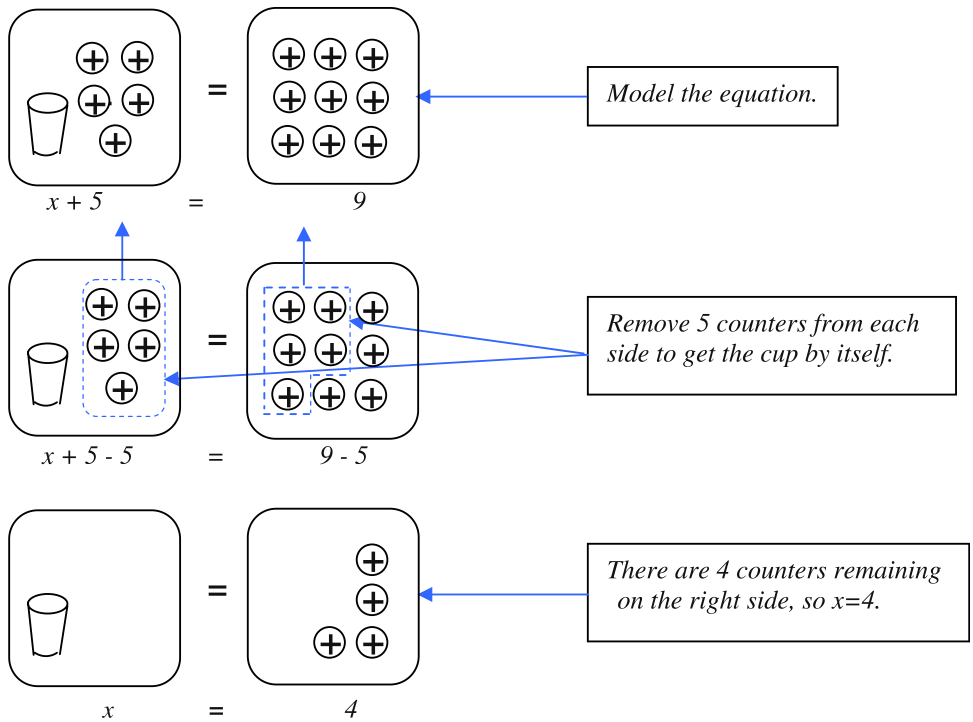


Fig. 4 Manipulatives are employed to show the steps of symbolic method of equation solving in Glencoe Mathematics (Bailey et al., 2006a, p. 337)

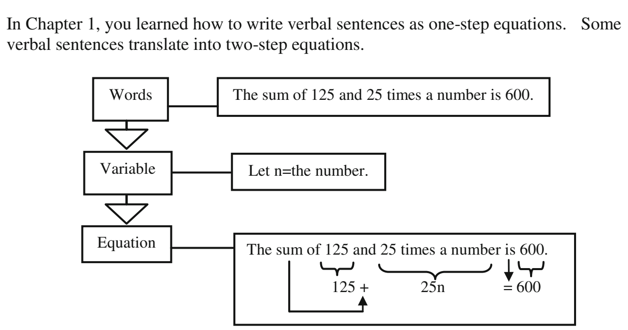


Fig. 5 A procedure from words to equations presented in Glencoe Mathematics (Bailey et al., 2006c, p. 478)

understand the meaning of relationships (functions) between two variables. Relationships and the concept of variable are introduced at the same time, at the beginning of the seventh grade. In fact, students are exposed to the concepts of "independent variable" and "dependent variable" before the unit on linear functions, which is not studied until the second half of the seventh grade.

CMP introduces the concept of linear relationships through the process of collecting empirical data, graphing the data and analyzing the resulting graphs. Empirical data pertinent to two variables is collected. For example, in the first investigation in *Moving Straight Ahead*, students record data on the drop height and the bounce height of a rubber ball. The data are then plotted on a coordinate graph. The graphical representation is used to illustrate the *relationship* between the two variables (bounce height and drop height). Students are asked to make observations about the data points on the graph and to make conjectures about their meaning. One conjecture that students might make is that the data points fall along a straight line on the graph. This gives students an experience that would demonstrate the CMP description of linear relationships: "relationships with graphs that are straight lines" (Lappan et al., 2002b, p. 3). To reinforce their understanding of this linear relationship, in problem 1 of the application section at the end of the investigation, students are given a table of drop height and bounce height data. They are asked to make a coordinate graph (see Fig. 6) and to analyze the data.

In later lessons, students are asked to generalize given tables of data into a "rule", describing the relationships between the given variables. At first, the rules are stated in words, but by the midpoint of the unit, students begin to state rules using symbols, as algebraic equations. The emphasis in this unit is on studying linear *relationships* by analyzing how two quantities (variables) covary when they are linearly related. The primary way in which CMP develops the idea of linear relationships is by having students study how changing one of the quantities in various linearly related situations affects the resulting tables, graphs and equations. So variables are still treated as quantities that change or vary. This interpretation of variable remains the same throughout the textbook series as it is in the first algebraic unit.

Exploring how the relationships between variables give rise to functions helps students reach a deeper understanding of both variables and functions. This approach to

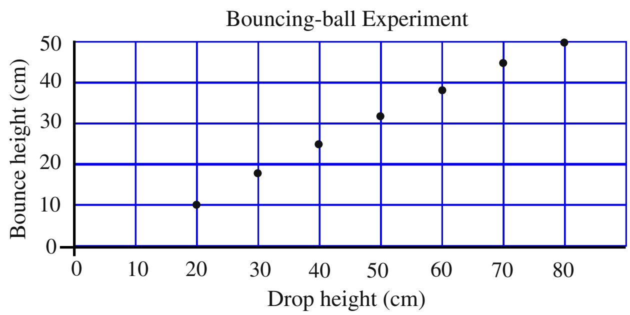


Fig. 6 A coordinate graph showing the answer to a CMP problem (Lappan et al., 2002b, p. 9)

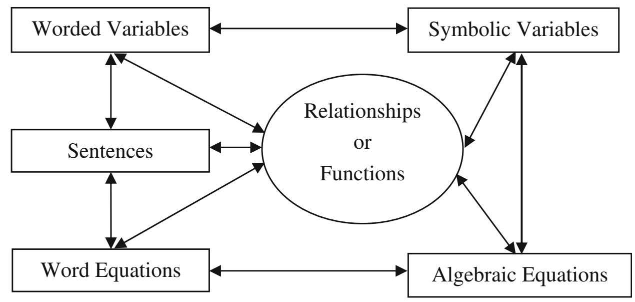


Fig. 7 The introduction of algebraic ideas in CMP

the development of the concept of variable in CMP reflects a central algebraic learning principle in the CMP curriculum: "As you study how variables are related, you are learning algebra" (Lappan et al., 2002a, p. 49).

After the concepts of variable and equation solving are introduced in CMP, a strong connection to linear relationships is maintained within the curriculum. We have already noted that after the symbolic method of solving equations is introduced in CMP, some of the equation-solving exercises are similar to the following: "find x if $326 = 4x$" and "find y in $y = x - 15$ if $x = 9.4$ and find x in $y = x - 15$ if $y = 29$" (Lappan et al., 2002b, p. 59). Indeed, it is a fact that all, but 4, of the 17 exercises at the end of that investigation require students to reflect on the relationship between the solution to one-variable linear equations and the corresponding two-variable linear equation. These types of exercises help develop students' facility for finding the values of independent and dependent variables in functions.

Figure 7 displays the connections among the ideas of variable, algebraic equations and linear function in the CMP curriculum. As we have seen, relationships (or functions) are at the center of the algebra curriculum, and word names for variables (worded variables) are used to analyze relationships between quantities long before symbolic names for variables (symbolic variables) are used to do so. Furthermore, word variables are used in sentences and word-based equations to describe relationships long before symbolic variables are introduced and used in algebraic equations to describe relationships.

4.4.2 *Linear functions in Glencoe Mathematics*

Glencoe Mathematics treats a function as a process of "starting with an input number, performing one or more *operations* on it, and getting an output number" (Bailey et al., 2006b, p. 177). In Glencoe Mathematics, the introduction of the concept of linear function is accomplished by means of a *function machine* with three key elements: input, output and operation. The operation, or rule, lies at the core of the function machine, while input and output are external to it. Actually, in this case, input and output should be called variables, but the term variable is not used to describe input or output at all until Course 2 where students learn that functions are often written as equations with two variables. Despite the curriculum's emphasis on input and output, it is interesting that, throughout Glencoe's three courses, we could not find any use of the terms *independent variable* and *dependent variable*.

By varying the values of input and output, it might be expected that Glencoe Mathematics intends to help students understand that variables are quantities that change. However, the main purpose of the *Function Machine* seems to be for students to experience the process of computing the output values from given input values and vice versa. That is, the development of the concept of function in Glencoe Mathematics emphasizes operations instead of the relationship between the variables. After the introduction of the *Function Machine* in Lesson 9-6a, Glencoe Mathematics introduces the concepts of function, function table and function rule in Lesson 9–6. For example, "the number of mosquitoes eaten by a bat is a function of the number of hours" (Bailey et al., 2006a, p. 362). Glencoe Mathematics provides a function table or problem situation and asks students to find the function rule. For example, "The second number is three more than the first number. Write an equation using x to represent the first number and y to represent the second number" (Bailey et al., 2006b, p. 180). This stresses the idea that "using a function is like finding a pattern" (Bailey et al., 2006a, p. 361). At this point, the variables are treated as being used to represent relationships.

In Lesson 9–7, students are expected to learn how to graph linear functions by making a function table, graphing ordered pairs on the coordinate plane and drawing the line that contains the points. Glencoe Mathematics does not always require students to draw enough points on the scatter plots to experience the linear relationships between two variables. For example, the line $y = 3x$ is drawn with just three points in the textbook, only the words "the points appear to lie on a line, draw the line that contains these points" are provided (Bailey et al., 2006a, pp. 366–367). We do not know why Glencoe Mathematics asks students to compute and plot only three points even though they have no prior understanding of linearity. However, it is possible that drawing more than three points might help students gain a better understanding of the linear relationship between two variables.

The effect of these kinds of treatments is that, unlike CMP, the concepts of symbolic variable, algebraic expression and algebraic equation in Glencoe Mathematics are connected in an almost linear way, as are the concepts of symbolic variable, algebraic expression and function. However, there is little or no emphasis on the connections between algebraic equations and functions (see Fig. 8). Similarly, Glencoe Mathematics presents equation solving and linear functions in a relatively independent way and few connections are made between these two algebraic ideas. Furthermore, Glencoe Mathematics devotes about twice as many pages to equation solving than to functions. Almost 8% of the pages (or 158 pages out of 2,029) from the entire Glencoe Mathematics curriculum focus on solving equations. In contrast, about 4% of the pages (or 82 pages out of 2,029) from the entire Glencoe Mathematics curriculum focus on functions. It very specifically categorizes equation-solving lessons in the textbook not only by the type of operations involved in solving the equations, but also by the number of steps involved in the equation-solving process. However, CMP devotes about 14 times as many pages to functions as it does to equation solving. Specifically, in CMP's seven algebraic units, only about 5% of the pages (or 25 pages out of 512) focus on solving equations, but about 72% of the pages (or 368 pages out of 512) focus on functions.

5 Discussion

In this section, we discuss, in turn, the results of each of the research questions that we have posed.

What are the differences between the learning goals for the concept of variable in the two curricula? We found that the learning goals for the concept of variable in these two curricula were consonant with the subsequent development

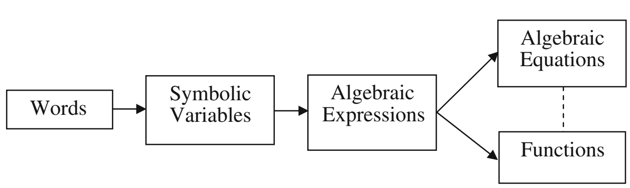


Fig. 8 The introduction of algebraic ideas in Glencoe Mathematics. *Note*: the *dashed line* between "Algebraic Equations" and "Functions" indicates the weak connection

of variable in each of the curricula. The CMP curriculum's learning goals are very explicit in their expectation: that students realize that variables are changing quantities used to represent relationships, and recognize variables in the real world. On the other hand, Glencoe's learning goals describe a variable as a placeholder or an unknown that is typically used in an equation-solving context.

How is the concept of variable introduced in the two curricula? We found significant differences in the intended treatment of the concept of variable in the CMP and Glencoe Mathematics curricula. In the CMP curriculum, a variable is a quantity that changes or varies, whereas in the Glencoe curriculum a variable is a symbol, usually a letter, used to represent a number. It would be hard to find two definitions of variable that are more dissimilar. We found that the choice of these two definitions of variable has significant and enduring implications not only for the development of the concept of the variable itself, but also for the development of both equation solving and function in these two curricula.

Variables in CMP have a dynamic quality since they are defined as quantities that change or vary. Accordingly, CMP materials use real-world situations to help students recognize, define and experience variables as quantities that change. As a result, the CMP curriculum uses variables primarily in the characterization of relationships between changing quantities (functions). Variables in Glencoe Mathematics, on the other hand, have a static quality since Glencoe defines variable as "…a symbol, usually a letter, used to represent a number." This definition gives rise to the use of variables as placeholders or unknowns, rather than as quantities that vary. In most cases, variables in Glencoe Mathematics are unknowns represented by symbols, empty parentheses, question marks or squares. Partly because these uses of variables are static, Glencoe Mathematics rarely employs real-world situations to help students understand the meaning of variables and the relationships between them.

Students' misunderstanding of the meaning of variables is well known (e.g., Booth, 1988; Knuth et al., 2005). Glencoe's definition and subsequent development of variable solely as a letter used to represent a number may have unintended consequences. As an example, one distressing confusion that may accompany this conception of variable is the position-in-the-alphabet misunderstanding. For example, if n is a variable that represents the number of sides of a figure each of length 2, when asked for the perimeter, students may reason that it is $2 \times 14 = 28$, rather than $2n$, since n is the 14th letter of the alphabet (Booth, 1988). The NCTM *Standards* caution "…most students will need extensive experience in interpreting relationships among quantities in a variety of problem contexts before they can work meaningfully with variables and symbolic expressions," (NCTM, 2000, p. 225). CMP's use of variables (both formally and informally) to represent real-world relationships, which is done well before the study of equation solving, is a way to prepare students to work meaningfully with symbolic expressions and to avoid confusions such as the position-in-the-alphabet misunderstanding.

How is the concept of variable related to the development of equation solving in the two curricula? Based on the definition of variable used in each of these curricula, the development of equation-solving abilities is very different in CMP and Glencoe Mathematics. Building on the development of variable as a quantity that changes, CMP emphasizes the use of variables in pairs to represent relationships between real-world quantities on coordinate graphs. This, in turn, leads to verbal descriptions of relationships between pairs of variables that represent real-world quantities, and by extension to two-variable algebraic equations that are inextricably bound to real-world situations. Therefore, to be true to this line of CMP development, single-variable equation solving is tied to the corresponding two-variable equations, and a procedure for solving such equations is developed by appealing to what makes sense in the accompanying reality. Glencoe Mathematics, on the other hand, builds on its definition of variable as "…a letter used to represent a number" by introducing and developing the concept of variable using a single variable, rather than a variable pair. This does not lead to the use of single-variable equations that are tied to real-world situations. So it is natural that Glencoe uses manipulative materials to develop a procedure for solving single-variable equations. Furthermore, it is natural to characterize the variables in these equations as "unknowns" whose values do not vary.

How is the concept of variable related to the development of linear functions in the two curricula? The organization and development of the algebraic units of the CMP curriculum focus on relationships and functions. Even before the concept of variable is formally introduced, CMP asks students to create tables and graphs of two quantities (variables), so they can learn to analyze and characterize relationships between two variables informally in word sentences, rather than in algebraic equations. Thus, the foremost aim regarding variables in CMP is that students come to understand that they are "used to represent relationships." Glencoe Mathematics, on the other hand, does not emphasize the use of variables to represent relationships. Glencoe Mathematics rarely employs real-world situations to help students understand the meaning of variables and the relationships between them. Many more pages are devoted to helping students learn to evaluate algebraic expressions and solve equations than to learning about functions. Although function machines are used to represent functions, they mainly show the process of how outputs are computed from inputs by operations.

6 Recommendations

For various reasons, we believe that it is unwise for a curriculum to emphasize one interpretation of the concept of variable to the exclusion of others. First, when a curriculum emphasizes that variables are letters, students may become confused when they realize that under different interpretations and circumstances, a variable may or may not be a letter (or symbol), and that a letter (or symbol) is not necessarily a variable. Throughout this paper, we have alluded to the impact that the former realization may have for students by drawing distinctions between the definition of variable in the CMP as a quantity (which can be expressed as a letter or a word) and in the Glencoe curriculum as a symbol. Students' confusion with the latter part of the statement can easily be imagined. For example, in both the CMP and Glencoe curricula, if the letter M represented the gender male, M would not be a variable because M does not name a quantity that varies, nor is M a symbol that is used to represent a number. However, there are many students who would consider M to be a variable.

Second, we believe that characterizing a variable primarily as an unknown may confuse students when they solve for "unknowns" in single-variable equations. Depending on the circumstance, students may encounter variables that are unknown or that they, in fact, may already know. For example, in the equation: $3x + 2 = 5$, we can tell students that the letter x is an unknown. However, students may realize that it is unknown only before they arrive at the number 1 as a solution to the equation. Obviously, as soon as the students solve the equation, the letter x is no longer an unknown. Thus, they might conclude that the letter x is just temporarily unknown. That is, from the moment students solve the equation, the letter x and the number 1 may well have overlapping images in their minds, in which case, for them, x is no longer unknown. In any case and no matter whether or not a student can solve the equation $3x + 2 = 5$, the student undoubtedly realizes that the value of the letter x that makes the equation true must be certain and unchangeable. So in this sense, if students consider the letter x to be both an unknown and a variable, they may become confused since the value of the unknown does not vary.

Third, a similar difficulty may arise when students work with two-variable functions. Depending on the situation, students may realize that the variables in functions expressed with algebraic equations in two variables may be either unknown or known. For example, in the linear function $3x + 2 = y$, the letters x and y can take on any of the values of the real numbers. In this case, compared to the unknowns in equations such as $3x + 2 = 5$, the variables are "really" unknown. However, the true power of the function rule lies in its ability to describe the relationship between the variables, and in a very real sense the question of whether the variables are known or unknown is not relevant. For example, in the function whose rule is $3x + 2 = y$, if the student randomly assigns a value such as 1 to x, the value of y cannot be randomly assigned. That is, y must be assigned the value 5 according to the function rule. More importantly, if the student increases the value assigned to x by a specific amount, the value assigned to y must be increased by three times that amount. In one sense, the function described by the algebraic equation $3x + 2 = y$ is just "borrowing" the form of an equation to represent a numerical relationship between the variables x and y that describes the covariation between the variables. The relative importance of the relationship between the variables, as compared to whether the variables are known or unknown, can be seen in the real-world example in which students conduct a survey of the heights and weights of eighth grade students in their school district. In this example, students should consider height and weight to be variables, because they are used to describe the relationship between the students' heights and weights. However, students who equate unknowns with variables may not consider height and weight to be variables, because they obviously know the heights and weights of all the subjects.

Finally, students may become confused when they realize that a variable can be a symbol, but a symbol is not necessarily a variable. A very typical example is that we use symbols π and e to represent the two irrational numbers 3.141… and 2.718…, respectively, but we never label them as variables. Another example is the symbol *3*. Students may say *3* is a number, but strictly speaking *3* is a very abstract symbol, because the student cannot find *3* in a real-world situation except if it involves mathematics. However, from one point of view, *3* is changeable because 3 can represent different groups of objects depending on the context. So, now the student may wonder, "If *3* is a symbol and changeable, why don't we call *3* a variable?" The key point is that the amount is not changing, but only objects are changing. Therefore, the major distinction between variables and symbols that are not variables is whether or not they represent quantities that change. In the example of the height–weight survey above, the reason students may use height and weight as the names of variables, even though they are not unknowns, is that they represent quantities that change. In fact, students may even substitute the names "height" and "weight" for h and w (or x and y) in an equation that describes the relationship between height and weight.

7 Conclusion

In summary, variable is a very complex concept because it involves different conceptions and meanings. Neither CMP nor Glencoe Mathematics clearly distinguishes among the various uses of variables. We believe it would be helpful if curricula distinguished truly varying variables from letters, unknowns and symbols by clearly pointing out the differences among them. When students begin to learn linear functions like $3x + 2 = y$, if they only hold the idea that x is an unknown as in the equation $3x + 2 = 5$, how can they understand that the values of x and y can change and then further explore the pattern of the change? On the other hand, if students think that variables always vary, how can they reconcile this with the use of variables to solve single-variable equations?

We agree with the NCTM that "An understanding of the meanings and uses of variables develops gradually as students create and use symbolic expressions and relate them to verbal, tabular, and graphical representations," (NCTM, 2000, p. 225). In line with this view, it is our opinion that, over time, it is desirable for a mathematics curriculum to clearly point out the different ways of using the word "variable" to avoid misunderstanding. We recommend that curricula should differentiate truly varying variable from letter, unknown and symbol, and we suggest that every algebra curriculum should highlight the interpretation that a variable represents a quantity that changes or varies. Therefore, we wonder if it is wise to have a single definition of the term variable, and we think that it may be more advantageous to point out to students that the term variable has many different uses and, therefore, has many different definitions, each depending on the purpose for which it is used.

References

Bailey, R., Day, R., Frey, P., Howard, A. C., Hutchens, D. T., McClain, K., et al. (2006a). *Mathematics: Applications and concepts (teacher wraparound edition), course 1*. Columbus, OH: The McGraw-Hill Company.

Bailey, R., Day, R., Frey, P., Howard, A. C., Hutchens, D. T., McClain, K., et al. (2006b). *Mathematics: Applications and concepts (teacher wraparound edition), course 2*. Columbus, OH: The McGraw-Hill Company.

Bailey, R., Day, R., Frey, P., Howard, A. C., Hutchens, D. T., McClain, K., et al. (2006c). *Mathematics: Applications and concepts (teacher wraparound edition), course 3*. Columbus, OH: The McGraw-Hill Company.

Booth, L. R. (1988). Children's difficulty in beginning algebra. In A. F. Coxford (Ed.), *The Ideas of algebra, K-12* (pp. 8–19). Reston, VA: NCTM.

Cai, J. (1995). A cognitive analysis of US and Chinese students' mathematical performance on tasks involving computation, simple problem solving, and complex problem solving. *Journal for Research in Mathematics Education Monographs Series*, *7*. Reston, VA: National Council of Teachers of Mathematics.

Cai, J. (2003). What research tells us about teaching mathematics through problem solving. In F. Lester (Ed.), *Research and issues in teaching mathematics through problem solving* (pp. 241–254). Reston, VA: National Council of Teachers of Mathematics.

Cai, J. (2004a). Introduction to the special issue on developing algebraic thinking in the earlier grades from an international perspective. *The Mathematics Educator (Singapore), 8*(1), 1–5.

Cai, J. (2004b). Why do U.S. and Chinese students think differently in mathematical problem solving? Exploring the impact of early algebra learning and teachers' beliefs. *Journal of Mathematical Behavior, 23*, 135–167.

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, S. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: A cross-cultural comparative perspective. *Zentralblatt fuer Didaktik der Mathematik (International Journal on Mathematics Education), 37*(1), 5–15.

Cai, J., Lo, J. J., & Watanabe, T. (2002). Intended treatment of arithmetic average in U.S. and Asian school mathematics textbooks. *School Science and Mathematics, 102*(8), 391–404.

Cai, J., & Moyer, J. C. (2006). *A conceptual framework for studying curricular effects on students' learning: Conceptualization and design in the LieCal Project*. Newark, DE: The University of Delaware.

Collins, W., Dritsas, L., Frey-Mason, P., Howard, A. C., McClain, K., Molina, D. D., et al. (1998a). *Mathematics: Applications and connections, course 1*. Columbus, OH: The McGraw-Hill Company.

Collins, W., Dritsas, L., Frey-Mason, P., Howard, A. C., McClain, K., Molina, D. D., et al. (1998b). *Mathematics: Applications and connections, course 2*. Columbus, OH: The McGraw-Hill Company.

Collins, W., Dritsas, L., Frey-Mason, P., Howard, A. C., McClain, K., Molina, D. D., et al. (1998c). *Mathematics: Applications and connections, course 3*. Columbus, OH: The McGraw-Hill Company.

Dowling, P. (1996). A sociological analysis of school mathematics text. *Educational Research in Mathematics, 31*, 389–415.

Ginsburg, M., & Clift, R. (1990). The hidden curriculum of preservice teacher education. In W. R. Houston & J. R. Flanders (Eds.), *Handbook of research on teacher education* (pp. 450–465). New York: Macmillan Publishing Company.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: Who gets an opportunity to learn what? *British Educational Research Journal, 28*(4), 567–590.

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education, 38*(4), 344–369.

Herman, R., Boruch, R., Powell, R., Fleischman, S., & Maynard, R. (2006). Overcoming the challenges: A response to Alan H. Schoenfeld's "What doesn't work". *Educational Researcher, 35*(2), 22–23.

Jackson, P. W. (Ed.). (1992). *Handbook of research on curriculum*. New York: Macmillan Publishing Company.

Janvier, C. (1996). Modeling and the initiation to algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 225–236). The Netherlands: Kluwer Academic Publishers, Dordrecht.

Kieran, C. (1988). Two different approaches among algebra learners. In A. F. Coxford (Ed.), *The ideas of algebra, K-12* (pp. 91–96). Reston, VA: NCTM.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence and variable. *Zentralblatt fuer Didaktik der Mathematik (International Journal on Mathematics Education), 37*(1), 69–76.

Kulm, G., Morris, K., & Grier, L. (1999). *Middle grades mathematics textbooks: A benchmark-based evaluation.* Washington, DC: American Association for the Advancement of Science: Project 2061. Available: http://www.project2061.org/publications/textbook/mgmth/report/default.htm. [04/15/09].

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002a). *Variables and patterns*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002b). *Moving straight ahead*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002c). *Thinking with mathematical models*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002d). *Looking for Pythagoras*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002e). *Growing, growing, growing*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002f). *Frogs, fleas, and painted cubes*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002g). *Say it with symbols*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002h). *Getting to know connected mathematics: An implementation guide*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002i). *Lesson planner for Grades 6, 7, and 8*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2002j). *Data about us*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006). *The shapes of algebra*. Upper Saddle River, NJ: Prentice Hall.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education, 31*, 234–241.

Moyer, J. C., Huinker, D., & Cai, J. (2004). Developing algebraic thinking in the earlier grades: A case study of the U.S. *The Mathematics Educator (Singapore), 8*(1), 6–38.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council (NRC). (2004). *On evaluating curricular effectiveness: Judging the quality of K-12 mathematics evaluations*. Washington, DC: The National Academies Press.

Ng, S. F. (2004). Developing algebraic thinking: A case study of the Singaporean primary school curriculum. *The Mathematics Educator (Singapore), 8*(1), 39–59.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., et al. (2002). *Why schools matter: A cross-national comparison of curriculum and learning*. San Fransisco, CA: Jossey-Bass.

Schoenfeld, A. H. (2006). What doesn't work: The challenge and failure of the works clearinghouse to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher, 35*(2), 13–21.

Schoenfeld, H. A., & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher, 81*(6), 420–427.

Senk, S. L., & Thompson, D. R. (2003). *Standards-based school mathematics curricula: What are they? What do students learn?*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed.), *The Ideas of algebra, K-12* (pp. 8–19). Reston, VA: NCTM.

Wheeler, D. (1996). Backwards and forwards: Reflections on different approaches to algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 317–325). The Netherlands: Kluwer Academic Publishers, Dordrecht.

Wu, H. (1997). The mathematics education reform: Why you should be concerned and what you can do. *American Mathematical Monthly, 104*, 946–954.