The Teaching of Equation Solving: Approaches in *Standards*-Based and Traditional Curricula in the United States

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ABSTRACT

This paper discusses the approaches to teaching linear equation solving that are embedded in a *Standards*-based mathematics curriculum (CMP) and in a traditional mathematics curriculum (Glencoe Mathematics) in the United States. Overall, the CMP curriculum takes a functional approach to teach equation solving, while Glencoe Mathematics takes a structural approach to teach equation solving. The functional approach emphasizes the important ideas of change and variation in situations and contexts. It also emphasizes the representation of relationships between variables. The structural approach, on the other hand, requires students to work abstractly with symbols, and follow procedures in a systematic way. The CMP curriculum may be regarded as a curriculum with a pedagogy that emphasizes predominantly the conceptual aspects of equation solving, while Glencoe Mathematics may be regarded as a curriculum with a pedagogy that emphasizes predominantly the procedural aspects of equation solving. The two curricula may serve as concrete examples of functional and structural approaches, respectively, to the teaching of algebra in general and equation solving in particular.

Key words: equation solving, mathematics curriculum, functional and structural approaches, standards

PURPOSE

When the topic of algebra in school mathematics is brought up, the first thing that usually comes to people's minds is equations and equation solving. Historically, equations have played a central role in the development of other aspects of mathematics, and in solving real-life problems. Even though there has been a major shift in the landscape of school mathematics in recent years (Chazan, 2008), learning to solve equations is still an essential element in the study of algebra (Mathematical Sciences Education Board, 1998). Unfortunately, many students still have a difficult time learning algebra, particularly learning the concepts and skills related to equations and equation solving (Kieran, 2007; Loveless, 2008; National Mathematics Advisory Panel, 2008). How can we teach students the fundamental ideas related to equations and equation solving that will provide a solid foundation from which to satisfy the quantitative demands of their future endeavors (Cai & Knuth, 2005; Kaput, 1999; RAND, 2003)?

The purpose of this paper is to compare the approaches to equation solving that are embedded in two types of middle school curricula used in the United States: "*Standards*-based" and "traditional." The *Standards*-based curriculum that we analyze in this paper is the Connected Mathematics Program (CMP). It was developed with the support of the National Science Foundation and designed to align with the reform-oriented principles recommended in the NCTM *Standards* (NCTM, 1989). The "traditional" curriculum that we analyze is Glencoe Mathematics. The National Science Foundation did not fund the development of Glencoe Mathematics. Although it professes to be *Standards*-based, Glencoe Mathematics generally is considered to be traditional in its approach, rather than reform-oriented. Our goal is not to evaluate these curricula. Instead, our intent is to acquaint the reader with the details of two alternate approaches to the teaching of equation solving, as well as the mathematical conceptions that underlie them.

In the past two decades, researchers have begun to explore new conceptions of school algebra (Heid, 1996; Kieran, Boileau & Garançon, 1996; Nemirovsky, 1996). Curriculum designers often disagree about the organizing themes that should be used to give coherence to algebra across the curriculum. Two ways to conceptually organize curricula written for school algebra are via functions and via structures (Algebra Working Group to NCTM, 1997). These two conceptions of school algebra are the basis for two popular approaches to the teaching of school algebra. The central mathematical concept of relationship underlies the functional approach, which has been advocated by many mathematics educators (Bednarz, Kieran, & Lee, 1996). Using the functional approach, the important ideas of change and variation that can be seen in various situations and contexts are used to organize algebraic concepts across the curriculum. The structural approach, on the other hand, looks beyond the potentially confounding aspects of real-world contexts. It focuses, instead, on procedures and on underlying structures and patterns. That is, the structural approach requires students to move away from contextual problems and develop the ability to generalize, work abstractly with symbols, and follow procedures in a systematic way.

In this paper, our analysis shows how these two approaches are implemented in *Standards*-based and traditional mathematics curricula, respectively. In addition, this paper provides some insights into the substance of the current reform effort in the United States, which has received widespread attention over the past decade.

BACKGROUND

Advocates of mathematics education reform often attempt to change classroom practice, and hence students' learning, by means of changes in curricula (Ball & Cohen, 1996). This is not a new development since, historically, curricula have been used to convey what students should learn as well as to simultaneously improve instruction. Therefore, an analysis of curricula can provide insights not only into different philosophies regarding mathematics learning, but also into different approaches to the teaching of mathematics (Cai, Lo, & Watanabe, 2002).

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published its first round of *Standards* documents (e.g. NCTM, 1989, 1991, 1995), which provided recommendations for reforming and improving K-12 school mathematics. These *Standards* documents not only specified new goals for school mathematics, but also specified major shifts in teaching mathematics, including movement toward:

- classrooms as mathematical communities--away from classrooms as simply collections of individuals;
- logic and mathematical evidence as verification--away from the teacher as the sole authority for right answers;
- mathematical reasoning--away from merely memorizing procedures;
- conjecturing, inventing, and problem solving--away from an emphasis on mechanistic answer-finding;
- connecting mathematics, its ideas, and its applications--away from treating math as a body of isolated concepts and procedures.

With extensive support from the National Science Foundation (NSF), a number of school mathematics curricula were developed and implemented to align with the recommendations of the *Standards*. The Connected Mathematics Program (CMP) is one of the *Standards*-based middle school curricula developed with funding from NSF. The CMP curriculum was designed to build students' understanding of important mathematics through explorations of real-world situations and problems. It is a complete middle-school mathematics program. Students using the CMP curriculum are guided to investigate important mathematical ideas and develop robust ways of thinking as they try to make sense and resolve problems based on real-world situations.

In this paper, we compare the approaches to teaching equation solving taken in the CMP curriculum to those taken in the more traditionally-based Glencoe *Mathematics: Concepts and Applications* curriculum (Bailey et al., 2006a, 2006b, and 2006c). The Glencoe curriculum is also a complete middle-school mathematics program. Students using the Glencoe curriculum are taught important mathematical skills and concepts principally by studying completely worked out examples with clear explanations that are paralleled by guided practice. As we have stated, mathematics educators consider the Glencoe curriculum to be traditional, rather than *Standards*-based. There is one book for each grade level in the Glencoe Mathematics. Unlike Glencoe Mathematics, the CMP curriculum consists of a number of unit booklets for each grade level.

The research reported here was part of a large project designed to longitudinally compare the effects of a *Standards*-based curriculum (CMP) to the effects of more traditional middle school curricula on students' learning of algebra. In the large project, *Longitudinal Investigation of the Effect of Curriculum on Algebra Learning* (LieCal)[1], we investigate not only the ways and circumstances under which CMP and other middle school curricula like Glencoe Mathematics can or cannot enhance student learning in algebra, but also the characteristics of the curricula that lead to student achievement gains (Cai, Moyer, Wang, & Nie, in press).

The LieCal project was conducted in 14 middle schools of an urban school district serving a diverse student population in the United States. Approximately 85% of the participants were minority students: 64% African American, 16% Hispanic, 4% Asian, and 1% Native American. Male and female students are about evenly distributed. One of the main purposes of the LieCal Project was to provide a profile of the intended treatment of algebra in the CMP

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[1] In 2006 and 2009, the CMP authors published revised editions of the CMP curriculum under the name CMP2. This article is based on the original CMP curriculum because the students in the LieCal project used CMP, not CMP2.

curriculum with a contrasting profile of the intended treatment of algebra in the non-CMP curricula (see Cai, et al., in press for details about the LieCal Project).

ANALYSIS FRAMEWORK

Cai (2004) developed an analysis framework that can be used to analyze the algebra strand of a curriculum. In particular, Cai et al (2005) used the framework to compare the algebra strands of elementary mathematics curricula used in China, Russia, Singapore, South Korea, and the United States. Using the framework involves analyzing the algebraic thinking features of a curriculum along three inter-related dimensions: (1) goal specification, (2) content coverage, and (3) process coverage.

Goal Specification. The goal specification dimension of the framework requires that the algebra-related goals of a curriculum be identified. Once identified, the goals are analyzed in light of the set of mathematical problems incorporated to help attain the goals. In particular, the analysis entails identification of the problem types and of the cognitive demand of the problems.

Content Coverage. For the content coverage dimension of the framework, the big ideas of algebraic thinking in a mathematics curriculum are identified. A big idea of algebraic thinking is an essential concept or technique for reasoning about quantities and relationships, such as, variables, proportional reasoning, patterns and relationships, equivalence of expressions, equation and equation solving, functions, change, representations, and modeling. These ideas are widely accepted as important in algebra. In this dimension of the framework, the development of the big ideas is also analyzed.

Process Coverage. Algebra is much more than just solving for x and y; instead, algebra is a way of thinking. Success in algebra depends on the ability to think in a variety of powerful ways that foster productive algebraic performance. When people think algebraically to solve problems, various habits of thinking come into play, such as Doing-Undoing, Building Rules to Represent Functions, and Abstracting from Computation (Driscoll, 1999). Curricula can serve to demystify algebra by providing activities that foster these sorts of thinking in students. In the process coverage dimension of the framework, how a curriculum is designed to develop algebraic-thinking habits is examined.

Cai et al. (2005) showed that the framework could be used successfully to identify the algebraic-thinking features of a curriculum and provide meaningful comparisons of algebra strands across curricula. In the LieCal Project, this framework has been adopted to analyze the algebra strands in both the CMP and the non-CMP curricula, and then use it to determine their unique features. In this paper, we report some results from the analysis. In particular, the focus of this paper is to convey how the concepts of equations and equation solving are treated in the CMP and Glencoe Mathematics curricula.

The results are presented in the following three sections: (1) introduction of equations, (2) introduction of equation solving, and (3) analysis of the types of problems that involve equations. Our goal is to provide a detailed account of the functional and structural approaches embedded in the CMP curriculum and Glencoe Mathematics, respectively, to teach equation solving.

INTRODUCTION OF EQUATIONS

In this section, we show how the CMP and the Glencoe Mathematics curricula incorporate the functional and structural approaches, respectively, into their introduction of the concepts of equation and equation solving. To understand the approaches to teaching equation solving in CMP and Glencoe Mathematics, it is necessary to understand how these two curricula define and introduce variable ideas.

Defining Variables

In a previous study (Nie, Cai, & Moyer, 2009), we compared the intended treatments of variable ideas in CMP and in Glencoe Mathematics. The learning goals in the CMP curriculum characterize variables as quantities used to represent relationships. In contrast, the learning goals in Glencoe Mathematics characterize variables as placeholders or unknowns.

In 7th grade, CMP formally defines a variable as "…a quantity that changes or *varies*. …" (Lappan et al., 2002a, p.7). We found that once CMP defines variables as quantities that change or vary, it uses them to represent relationships.However, CMP employs an informal use of the concept of variable long before formally defining variable in the 7th grade. That is, the CMP 6th grade curriculum teaches students to describe relationships using quantities that vary.

Glencoe Mathematics, on the other hand, formally defines a variable in 6th grade as a symbol (or letter) used to represent a number. It treats variables predominantly as placeholders by using them to represent unknowns in expressions and equations.

Defining Equations

With its emphasis on relationships, CMP clearly approaches the concept of *variable* functionally. On the other hand, Glencoe Mathematics' focus on variable as a symbol points toward its structural approach. It is not surprising, therefore, that the concept of *equation* is defined functionally in CMP, but structurally in Glencoe Mathematics.

Functional approach in the CMP curriculum. In CMP, the concept of equation is functionally based. This approach is a natural extension of CMP's development of the concept of variable described above and is based on the guiding principle that expressing relationships between variables is at the heart of algebra. Representations that express these relationships are introduced incrementally. At the beginning of 6th grade, relationships between quantities are expressed using graphs and tables rather than via algebraic equations. CMP does not even introduce algebraic equations when it formally defines the term *variable* in the 7th grade unit *Variables and Patterns* (Lappan et al., 2002a). Instead, CMP simply prepares for the eventual introduction of equations as descriptors of relationships by introducing the terms *independent variable* and *dependent variable*, which are vital in the language of equations and functions.

Equations themselves are introduced later through contextual examples that give rise to formulas or "*rules"* that model a given "real life" context. An instance of this can be found on page 49 of *Variables and Patterns*:

$$\textit{circumference} = \pi \times \textit{diameter}$$

which is later referred to as an example of an "*equation*" or a "*formula*." The idea of an equation is developed subsequently in activities requiring students to:

- Generate equations ("rules") from graphs and tables that they make—
  - Example: "Look for patterns in the table and graph that help you calculate the distance traveled for any given time. Write a rule, using words, that explains how to calculate the distance traveled for any given time. Use symbols to write your rule as part of an equation" (p. 51).
- Compare and contrast similarities and differences among different generated equations—
  - Example: "How are the rules for calculating distance for the three speeds similar? How are they different?" (p. 52).
- Derive contextualized meaning from equations—
  - Example: "After arriving in Philadelphia, Malcolm took the interstate home. He wrote the equation d = 60t to represent his trip home. Explain this equation in words" (p. 52).
- Match given scenarios to appropriate equations—
  - Example: "Theo's father has a van he will let the students use at no charge. [Bike rental costs $30 per person, and food and camp costs $125 per person.] Which of these equations represents the total cost if they use his van? (a) C = 125 + 30; (b) C = 125n + 30n; (c) C = 155; (d) C = 155 + n"(p. 53)

It is apparent from the above examples that the emphasis in CMP is on using equations to describe real-world situations. Rather than seeing equations simply as objects to manipulate, students are shown that equations often describe relationships between varying quantities that arise from meaningful, contextualized situations. Clearly, this view of *equation* fits within the framework of a functional approach (Bednarz et al., 1996).

The functional approach to introducing the concept of equation is also apparent in CMP's emphasis on multiple ways of representing equations. In the unit *Variables and Patterns*, students study the graphs of various equations *using a graphing calculator.* In addition, students study tables corresponding to various equations. The intention is to help students to understand relationships among the symbolic, graphical, and tabular representations of equations. It is instructive to note that in CMP, students' initial exposure to equations does not involve solving equations.

Structural approach in the Glencoe Mathematics Curriculum. In this section, we show how Glencoe Mathematics' definition of variable as a symbol develops naturally into the use of "naked" equations, and an emphasis on procedures for solving equations. These are all hallmarks of a structural focus.

Lesson 1-7, entitled "Algebra: Solving Equations," of the Glencoe Mathematics 6th grade textbook (Bailey et al., 2006a), introduces equations shortly before defining them. The lesson begins with a Hands-On Mini-Lab in which students represent single-variable equations on a balance scale. On the scale, a paper cup represents the variable (placeholder), and groups of centimeter cubes represent numerical constants. Students are told that when the amounts on each side of the scale are equal, the scale is balanced. The students place 3 cubes and a cup on one side of the scale and 8 cubes on the other. Then they are instructed to replace the paper cup with centimeter cubes until the scale balances. By way of practice, the students use the scale to model four other equations and find the number of centimeter cubes needed to balance the scale for each. Neither the word *equation* nor the word *solve* is used in the Mini-Lab.

After the Mini-Lab, in Lesson 1-7 itself, Glencoe defines an equation as "…a sentence that contains an equals sign" (p. 34). By way of illustration, the book then provides examples of number sentences (p. 34):

$$2 + 7 = 9 \qquad 10 - 6 = 4 \qquad 4 = 5 - 1$$

However, the text does not explicitly relate these examples to the Mini Lab. Therefore, it is conceivable that these examples actually reinforce erroneous interpretations of the equals sign as a symbol that signifies the result of a computation (e.g., when 2 is added to 7, the result is 9) (Kieran, 1981). As a consequence, some students may continue to mistakenly believe that an equals sign says "Write the answer or result of the indicated computation."

Immediately following these arithmetic-based examples of equations, the text illustrates equations that contain variables:

$$2 + x = 9 \qquad 4 = k - 6 \qquad 5 - m = 4$$

Students are told that the way to solve an equation is to replace the variable with a value that results in a true sentence. It is worth noting that, at this point in the lesson, the text does not refer to the hands-on Mini-Lab, nor does it make any explicit reference to how the notion of balance relates to equations or equation solving.

An important equation-solving technique in Glencoe Mathematics is to use mental math, e.g. "THINK 12 equals 5 plus what number?" (p. 35). After solving an equation mentally, students are asked to graph the solution on a number line. Almost all of the equations in chapters 1-8 of the 6th grade text can be solved using mental addition or subtraction. Equation solutions that require multiplication or division are first introduced in Lesson 9-4, Solving Multiplication

Equations. Also, the Addition Property of Equality and the Subtraction Property of Equality are formally stated and used in Chapter 9, and the Multiplication and Division Properties of Equality are used, but not formally stated.

Glencoe Mathematics utilizes a spiral technique to introduce equations. In 7th grade, for example, equations are introduced in a similar way as in the 6th grade. However, rather than waiting until later to introduce equations that require the use of mental multiplication or division, the 7th grade text introduces equations using examples that are solved using either mental addition, subtraction, multiplication, or division:

$$p - 5 = 20 \qquad 8 = y \div 3 \qquad 7h = 56$$

In Grade 7, as in grade 6, the concept of *equation* is also introduced using number sentences as examples. However, the number sentences used are quite different from what they were in Grade 6. In the Grade 7 examples, it is more evident that the equals sign does not signify the result of a computation, instead it represents equality or equivalence, e.g. "4 + 3 = 8 -1", "3(4) = 24 / 2" and "17 = 13+ 2+ 2" (Bailey et al., 2006b, p. 24).

The structural approach to equations and equation solving can also be seen in its use of a "dictionary" of common terms associated with the four main operations of mathematics. The dictionary is presented in Lesson 4-1 of the 7th grade text (Bailey et al., 2006b, p.150) and Lesson 1-7 of the 8th grade text (Bailey et al., 2006c, p.39), both titled "Writing Expressions and Equations." In a corresponding 8th grade lesson, Glencoe Mathematics builds on this structural approach by formally introducing a 3-step process for writing expressions and equations. Step 3 of the process relies on the use of the dictionary (Bailey et al., 2006c, p.39).

(Insert Table 1 about Here)
(Insert Figure 1 about Here)

In order to make the transition from writing algebraic expressions to writing equations, Glencoe Mathematics tells students that "When you write a verbal sentence as an equation, you can use the equals sign (=) for the words *equals* or *is*" (Bailey et al., 2006b, p. 151). After showing how to write a phrase as an expression, Glencoe Mathematics provides examples showing how to write sentences as equations (Bailey et al., 2006b, pp. 150-151), as shown in Figure 2.

(Insert Figure 2 about Here)

In addition, the reverse situation is described: "write a verbal sentence that translates into the equation n + 5 = 8" (Bailey et al., 2006b, p. 151). Students must think critically to come up with a real life scenario that matches the equation given. This is intended to help them see the value of using algebraic equations.

INTRODUCTION TO EQUATION SOLVING

CMP and Glencoe Mathematics use functional and structural approaches, respectively, to introduce equation solving, consistent with the approaches they used to define equations.

Introduction to Equation Solving in the CMP Curriculum

In the CMP curriculum, equations are introduced as descriptors of relationships between variables. In the second algebra unit in 7^{th} grade—*Moving Straight Ahead* (Lappan et al., 2002b), the topic of linear equations and functions is formally introduced as "linear *relationships."* This way of introducing linear equations and functions is a strong indication of CMP's heavy emphasis on the conception of algebra as the study of relationships.

Students begin this unit on linear equations, not by solving or analyzing any given equations, but by collecting empirical data pertinent to two quantities (variables), height of a rubber ball when released and height of its bounce. Students graph the data on a coordinate graph and look for a *relationship* between the variables. In addition, students analyze graphs, tables, and equations that represent relationships between pairs of variables taken from a variety of situations, and they compare these representations with one another. For example, in one of the activities students make tables, graphs, and equations for the following situation:

> *Ms. Chang says that some sponsors might ask the students to suggest a pledge amount. The class wants to agree on how much they will ask for. Leanne says that $1 per mile would be appropriate. Gilberto says that $2 per mile would be better because it would bring in more money. Alana points out that if they ask for too much money, not as many people will want to be sponsors. She suggests that they ask each sponsor for a $5 donation plus 50 cents per mile* (p. 19).

Students are asked to examine how increasing the amount pledged per mile affects each of the representations, how the $5 donation is represented within each of the representations, and so forth.

CMP introduces equation solving within the context of discussing linear relationships. The initial treatment of equation solving does not involve symbolic manipulation, as found in most traditional curricula. Instead, CMP attempts to introduce students to linear equation solving by making visual sense of what it means to find a solution. Its premise is that a linear equation in one variable is, in essence, a specific instance of a corresponding linear relationship (or equation in two variables). It relies heavily on the context in which the equation itself is situated, and on the use of a graphing calculator.

Students are first given various equations in *two variables*, each modeling a "real-world" context (e.g., $A = 5 + 0.5d$, where A represents dollars owed and d represents number of miles walked). Then various questions are asked in which either a value for A or d is given (e.g., $17 is owed, or someone walked 28 miles), and subsequently, the other value must be found. Students solve this problem by first graphing the original equation in two variables on a graphing calculator, then finding the value of either the given dependent or the given independent variable on the graph, and finally reading off the other value as the solution. This graphical approach is intended to help students understand the *meaning of a solution* to a linear equation and the *process of solving an equation* in one variable.

After equation solving is introduced graphically, the symbolic method of solving linear equations is finally broached (p. 55). It is introduced within a single contextualized example, where each of the steps in the equation solving process is accompanied by a narrative that demonstrates the connection between the procedure and the real-life situation (see Table 2). In this way, CMP justifies the equation-solving manipulations through contextual sense-making of the symbolic method. That is, CMP uses real-life contexts to help students understand the meaning of each step of the symbolic method, including why inverse operations are used.

(Insert Table 2 about Here)

Later, the CMP 8th grade unit *Say It with Symbols* (algebraic reasoning) (Lappan et al., 2002c), revisits solving linear equations. CMP uses a functional approach to re-introduce the topic of solving linear equations. Rather than beginning with an equation to solve, such as the equation, 100 + 4x = 25 + 7x, CMP poses a context that generates the two equations y = 100 + 4x and y = 25 + 7x (the two sides of the original equation), and asks students to use graphs and tables to find the value of x for which the two y-values are equal. Once the students have explored two examples that use this approach, CMP introduces the conventional symbolic (manipulation) approach, first through a short exposition, and then through a few practice problems.

<u>Introduction to Equation Solving in Glencoe Mathematics</u>

In the Glencoe Mathematics curriculum, contextual sense-making is not used to justify the equation-solving steps, as it is in the CMP curriculum. Rather, Glencoe Mathematics first introduces equation solving by finding a number to make an equation a true statement. In fact, *solving* an equation is described as replacing a variable with a value (called the *solution*) that makes the sentence true (Bailey et al., 2006a, p. 34) (See Figure 3).

(Insert Figure 3 about Here)

In 6th grade, Glencoe formally introduces equation solving with inverse operations by way of an activity that uses a cup to stand for an unknown (see Figure 4 taken from Bailey et al., 2006a, p. 337). The cups and counters used as manipulatives in the activity are simply exact representations of the symbols. That is, the manipulatives illustrate each step of the symbolic manipulations.

(Insert Figure 4 about Here)

Solving the equation using manipulatives, as shown above, is referred to as "Method 1." An illustration of the method typically is placed adjacent to an example showing a corresponding solution that uses "Method 2," the symbolic method. In this way, Glencoe Mathematics illustrates how each manipulative step is comparable to a symbolic step in a solution based on algebraic properties of equality, which is shown vertically. Figure 5 shows an example of how to use Method 2 to solve a one-step equation.

(Insert Figure 5 about Here)

Effort has been made in Glencoe Mathematics to connect tables, formulas, and graphs. For example, the 6th grade curriculum shows the table of values corresponding to an expression as well as the corresponding ordered pairs (see Figure 6). Then the ordered pairs are graphed, and the graph is described (Bailey et al., 2006a, p. 322)

(Insert Figure 6 about Here)

In the 8th grade curriculum, Glencoe replaces cups and counters with algebra tiles in order to illustrate how to solve more complicated equations (Bailey et al., 2006c, p. 468). An example is shown in Figure 7, in which these tools are used to model and solve the equation $x + 3 = -2$ from Chapter 10 (Bailey et al., 2006c, p. 468).

(Insert Figure 7 about Here)

ANALYSIS OF MATHEMATICAL PROBLEMS

The mathematical problems included in a curriculum can reveal not only the instructional goals and principles that the authors espouse, but also the learning opportunities they provide for students. Problems serve to direct students' attention to particular aspects of content, as well as their ways of processing information (NCTM, 2000).

Cognitive Demand of Problems in CMP Curriculum and Glencoe Mathematics

We identified all the mathematical problems involving equations in both the CMP and Glencoe Mathematics curricula. The vast majority of the problems in the CMP curriculum that involve equations are problems that require extended investigations, thus there are fewer CMP equation-related problems than Glencoe Mathematics problems. There are a total of 666 problems involving equations in the CMP curriculum and 2,453 problems in Glencoe Mathematics. We classified all the equation-related problems into four categories according to the levels of cognitive demand (Stein et al., 1996): (1) memorization, (2) procedures without connections, (3) procedures with connections, and (4) doing mathematics. Figure 8 shows the percentage distribution of equation problems in each of the four cognitive levels in both curricula. The distributions of the problems in the four cognitive levels are significantly different (χ^2 (3) = 1361.71, $p < .0001$). The CMP curriculum includes significantly larger percentages of problems at the two higher levels of cognitive demand (Procedure with Connections or Doing Mathematics) than does Glencoe Mathematics ($z = 36.25$, $p < .0001$). In contrast, there is a significantly larger percentage of the problems in the two lower levels of cognitive demand (Procedure without Connections or Memorization) in Glencoe Mathematics than in the CMP curriculum.

(Insert Figure 8 about here)

Types of Problems Involving Linear Equations

In both the CMP and Glencoe Mathematics curricula, the vast majority of the equation problems involve linear equations. Thus we further classified problems involving linear equations in the CMP and Glencoe curricula into three categories:

(1) One equation with one variable (1equ1va)--e.g., $2x + 3 = 5$;

(2) One equation with two variables (1equ2va)--e.g., $y = 6x + 7$;

(3) Two equations with two variables (2equ2va)--e.g., the system of equations $y = 2x + 1$ and $y= 8x + 9$.

Table 3 shows the percentage distribution of the problems involving linear equations in the two curricula. These two distributions are significantly different (χ^2 (2) = 1262.0, $p < .0001$). The CMP curriculum includes a significantly greater percentage of "one equation with two variables" problems than the Glencoe curriculum ($z = 35.49$, $p < .0001$). On the other hand, the Glencoe curriculum includes a significantly greater percentage of "one equation with one variable" problems than the CMP curriculum ($z = 34.145$, $p < .0001$). These results resonate with the findings that we reported above. Namely, that the CMP curriculum emphasizes an understanding of the relationships between the variables of equations, rather than an acquisition of the skills needed to solve them. In fact, of the 402 equation-related problems in the CMP curriculum, only 33 of them (about 8% of the linear equation-solving problems) involve decontexualized symbolic manipulations of equations. However, Glencoe Mathematics includes 1,550 problems involving decontexualized symbolic manipulations of equations (nearly 70% of the linear equation solving problems in Glencoe Mathematics).

(Insert Table 3 about here)

Glencoe Mathematics not only incorporates many more linear equation-solving problems into the curriculum, but it also carefully sequences them based on the number of steps required to solve them. Of 2,339 problems involving linear equations, 1,218 of them (over 50%) are one-step problems like, $x + b = c$, $ax = c$ or $x = a*b$. Nearly 700 of them (or about 30%) are two-step problems, like $ax + b = c$ or $x/a = b/c$. Only a small proportion of the linear equations involve three steps or more, like $ax + bx + c = d$ or $ax +b = cx + d$. Each grade of Glencoe Mathematics includes one-step, two-step, and three-plus-step problems involving linear equations. As the grade level increases, however, the Glencoe Curriculum provides increasingly more comprehensive procedures, suitable for solving all forms of linear equations. or the problems involving one equation with two variables in the CMP curriculum, over 50% of them are in $y = ax + b$ form ($a \neq 0$, $a \neq 1$ and $b \neq 0$). Over 30% of them are in $y = x + b$ form ($b \neq 0$). The rest of them are either in $y = c$ form or $y = ax$ form ($a \neq 0$).

CONCLUSION

How can we effectively teach algebra? In this paper, we discussed the approaches to teaching linear equation solving that are embedded in a *Standards*-based mathematics curriculum (CMP) and in a traditional mathematics curriculum (Glencoe Mathematics). Overall, the CMP curriculum uses a functional approach to teach equation solving, while Glencoe Mathematics uses a structural approach to teach equation solving. The functional approach emphasizes the important ideas of change and variation in situations and contexts. It also emphasizes the representation of relationships between variables, which many mathematics educators feel is at the heart of school algebra. The structural approach, on the other hand, avoids contextual problems in order to concentrate on developing the abilities to generalize, work abstractly with symbols, and follow procedures in a systematic way. These abilities are considered by other mathematics educators to be at the heart of school algebra.

The CMP curriculum includes very few equation-solving problems that require the use of conventional symbolic manipulations. Using a functional approach, the CMP curriculum defines variables as quantities that change or vary and that are used to represent relationships. As a natural extension, the CMP curriculum introduces equations as a way of studying relationships. The intent of the CMP curriculum is that students learn to view equations as instruments to describe real-world situations, rather than simply as objects to manipulate. Correspondingly, equation solving is introduced within the context of discussing linear relationships. Thus, the vast majority of the linear equations involve two variables. A two-variable graph is used to introduce students to the *meaning of a solution* to a linear equation in one variable as well as to the *process of solving an equation* in one variable. CMP uses real-life contexts to help students understand the meaning of each step of the symbolic method of equation solving.

Glencoe Mathematics, on the other hand, formally defines a variable as a symbol (or letter) used to represent a number. It treats variables predominantly as placeholders by using them to represent unknowns in expressions and equations. Maintaining their structural approach, Glencoe Mathematics defines an equation as a sentence that contains an equals sign. In the Glencoe Mathematics curriculum, contextual sense making is not used to justify the equation-solving steps. Rather, Glencoe Mathematics introduces equation solving as a process to find a number that makes the sentence a true statement.

Equation solving is highly conceptual and procedural in nature. It is highly conceptual since it involves an understanding of mathematical relationships. It is highly procedural since it involves the steps to find solutions to equations. The primary focus of the functional approach in the CMP curriculum is related to conceptual understanding, and the primary focus of the structural approach in Glencoe Mathematics is related to procedural understanding. In this sense, the CMP curriculum may be regarded as a curriculum with a pedagogy that emphasizes predominantly the conceptual aspects of equation solving, while Glencoe Mathematics may be regarded as a curriculum with a pedagogy that emphasizes predominantly the procedural aspects of equation solving.

The results reported in this paper not only show the unique features of the CMP and the Glencoe Mathematics curricula, but also present the CMP and the Glencoe Mathematics curricula as concrete examples of functional and structural approaches, respectively, to the teaching of equation solving.

It is important to indicate that any curriculum has a complex relationship to what actually occurs in classrooms. In this paper, the focus of our discussion has been on the intended treatment of linear equation solving in the CMP and Glencoe Mathematics curricula. Elsewhere, we have reported that different profiles of classroom instruction and student learning occur when using CMP and non-CMP curricula like Glencoe Mathematics (Cai et al., in press). Generally speaking, we found that the type of curriculum that teachers use has a significant effect on teaching that they do. For example, corresponding to the curricular differences we reported in the percentages of high-level problems in the two types of curricula, we found a significant difference between the two percentage distributions of the cognitive demand of the instructional tasks implemented in the CMP and non-CMP classrooms ($\chi^2(3) = 219.45$, $p < .0001$). This is due to the fact that the CMP teachers implemented a larger percentage (45%) of high cognitive demand tasks (procedures with connection or doing mathematics) than the non-CMP teachers (10%) ($z = 14.07$, $p < .0001$). Equivalently, from the opposite point of view, the significant difference between the two distributions is due to the fact that the non-CMP teachers implemented a larger percentage of low cognitive demand tasks (procedures without connection or memorization) (90%) than the CMP teachers (55%).

Based on our analysis of the curricula, we have argued in this article that the CMP curriculum may be regarded as a curriculum with a pedagogy that emphasizes predominantly the conceptual aspects of equation solving, while Glencoe Mathematics may be regarded as a curriculum with a pedagogy that emphasizes predominantly the procedural aspects of equation solving. In line with this result, our LieCal investigation of classroom instruction found that CMP teachers emphasized the conceptual aspects of learning significantly more often than the non-CMP teachers ($t = 12.40$, $p < .001$). On the other hand, non-CMP teachers emphasized the procedural aspects of learning significantly more often than the CMP teachers ($t = 10.43$, $p < .001$).

We also found that on open-ended tasks assessing conceptual understanding and problem solving, the growth rate for CMP students over the three years (grades 6-8) is significantly greater than that for non-CMP students (Cai et al., in press). In fact, our analysis using Growth Curve Modeling showed that over the three middle school years, the CMP students' scores on the open-ended tasks increased significantly more than the non-CMP students' scores ($t = 2.79$, $p < .01$). On the other hand, CMP and non-CMP students showed similar growth over the three middle school years on the multiple-choice tasks assessing computation and equation solving skills.

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Table1. Some common terms associated with the four operations (from Glencoe Mathematics)

Addition or Subtraction				Multiplication or Division			
plus	increased by	minus	subtract	times	each	divided	rate
sum	in all	less	decreased by	product	of	quotient	ratio
total	more than	less than	difference	multiplied	factors	an, in, or per	separate

Table 2. An Example of Equation Solving in CMP (p. 55)

Thinking	Manipulating the symbol
"I want to buy a CD-ROM drive that costs \$195. To pay for the drive on the installment plan, I must pay \$30 down and \$15 a month."	195 = 30 + 15N
"After I pay the \$30 down payment, I can subtract this from the cost. To keep the sides of the equation equal, I must subtract 30 from both sides	195 – 30 = 30 – 30 + 15N
"I now owe \$165 which I will pay in monthly installments of \$15."	165 = 15N
"I need to separate \$165 into payments of \$15. This means I need to divide it by 15. To keep the sides of the equation equal, I must divide both sides by 15."	165/15 = 15N/15
"There are 11 groups of \$15 in \$165, so it will take 11 months."	11 = N

Table 3. Percentage Distribution of Problems Involving Linear Equations in CMP and Glencoe curricula

	1equ1va	1equ2va	2equ2va
CMP(*n*=402)	5.72	93.03	1.24
Non-CMP(*n*=2339)	86.19	11.67	2.14

Figure 1. A Three-Step Process for Writing Algebraic Expressions in Glencoe Mathematics

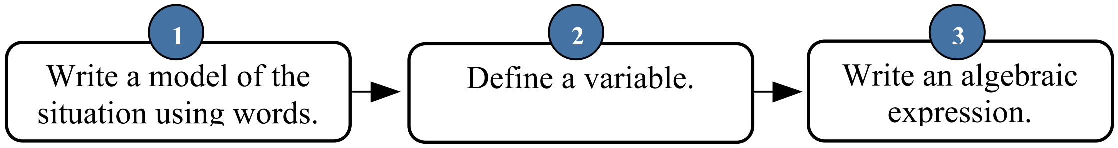


Figure 2. <u>Sample Examples of Translating Written Sentences into Equations in Glencoe Mathematics</u>

Example 1: Write the phrase *five dollars less than Jennifer earned* as an algebraic expression

Words	Five dollars less than Jennifer earned.
Variable**0**	Let *d* represent the number of dollars Jennifer earned.
Expression	d - 5

Example 2: **Sentence** **Equation**

Five more than a number is 20 — $n + 5 = 20$

Example 3: Three times Bill's age equals 12 — $3a = 12$

Example 4: It is estimated that 12.4 million pounds of potato chips were consumed during a recent super Bowl. This was 3.3 million pounds more than the number of pounds of tortilla chips consumed. Write an equation that models this situation.

Words	Potato chips were 3.1 million more than tortilla chips.
Variable**0**	Let t = number of million pounds of tortilla chips.
Expression	$12.4 = 3.1 + t$.

Figure 3. Meaning of Solving an Equation in Glencoe Mathematics

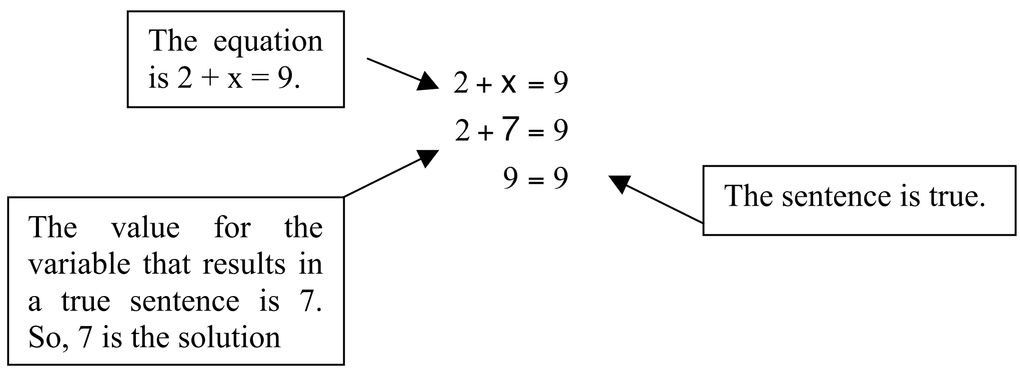


Figure 4. Introduces Equation Solving with Inverse Operations in Glencoe Mathematics

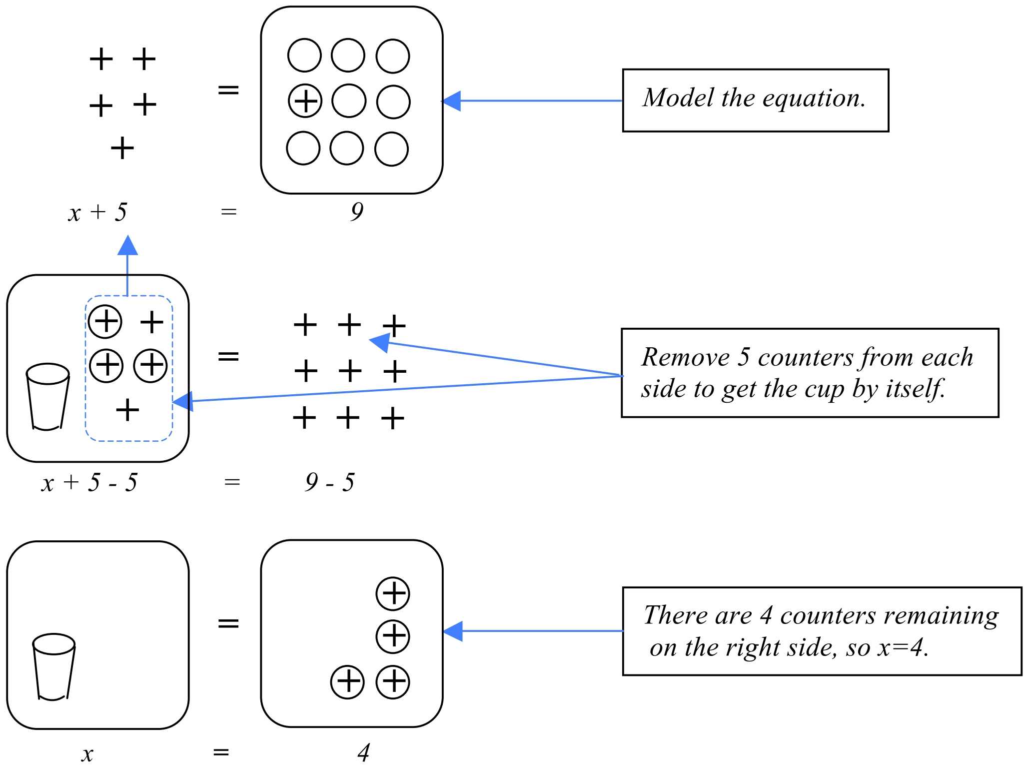


Figure 5. Symbolic Representation of Solving an Equation in Glencoe Mathematics

$x + 5 = 9$ Write the equation.

Subtract 5 from each side to undo x plus 5.

$$\begin{aligned} x + 5 &= 9 \\ -5 &= -5 \end{aligned}$$

Subtract 5 from each side.

$x + 0 = 4$ $5 - 5 = 0$, $9 - 5 = 4$

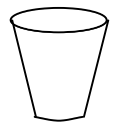
$x = 4$

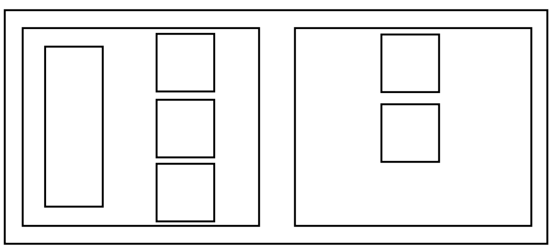
Figure 6. Connecting a Table, Formula, and Graph in Glencoe Mathematics

In basketball, each shot made from outside the 3-point line scores 3 points. The expression 3*x* represents the total number of points scores where *x* is the number of 3-point shots made. …List the ordered pairs (3-point shots made, total number of points) for 0, 1, 2, and 3 shots made….Make a table, Graph the ordered pairs … then describe the graph.

x (shots)	3*x*	*y* (points)	(*x*,*y*)
0	3(0)	0	
1	3(1)	3	(1,3)
2	3(2)	6	(2,6)
3	3(3)	9	(3,9)

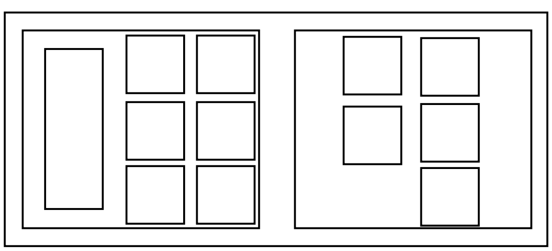
Figure 7. Algebra Tiles for modeling Equation Solving

Type of Model	Variable *x*	Integer 1	Integer -1
Cups and Counters		+	−
Algebra Tiles	***x***	1	- 1



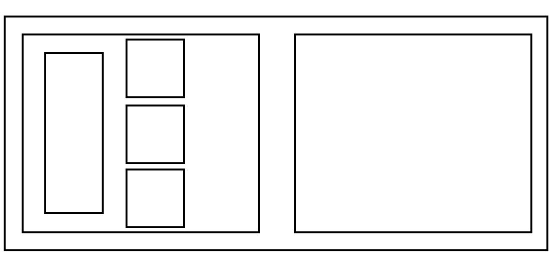
x + 3 = -2

Model the equation.



x + 3 + (-3) = -2 + (-3)

Add three -1 tiles to each side of the mat. The left side now contains zero pairs.



x = -5

Remove the zero pairs from the left side. The *x*-tile is now isolated. There are 5 negative tiles on the right side of the mat.

Therefore, $x = -5$. Since $-5 + 3 = -2$, the solution is correct.

Figure 8. Cognitive Demand of Equation Problems in CMP Curriculum and Glencoe Mathematics