• Please read each question carefully.
• Please show your work clearly on the sheets provided.
• Answers without justification will be assigned zero credits. Explanation provided after the exam will be ignored.
• Each of the problems in the exam has appeared in class/text or previous tests. Do not indulge yourself on a particular problem for too long.

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I. a) Write down the Maclaurin series for \( \ln(1 + x) \) at \( a = 0 \). You need to write down the general \( n \text{th} \) term of the series. If you cannot remember the formula, you may derive the coefficients again by computing the derivatives first.

\[ f(x) := \ln(1 + x), \text{ by repeated differentiation, we can find that} \]
\[ f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1 + x)^n} \]

for \( n \geq 1 \).

\[ c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n} \cdot \]

Also \( c_0 = f(0) = \ln(1) = 0 \). Therefore \( \ln(1 + x) = x - x^2/2 + x^3/3 + \cdots \). \( \frac{1}{1+x} \) and then integrate the series term by term.

b) Using appropriate methods discussed in class, find all the values of \( x \) that Maclaurin series converges at.

\[ \text{Ans.} \]
\[ \text{The interval is } (-1, 1) \text{ which can be obtained by ratio test. Another way is to realize the geometric series has the interval of convergence: } |x| < 1. \]

At the end point \( x = 1 \), you obtain the alternating harmonic series, therefore the series converges. At \( x = -1 \), you have the regular divergent harmonic series. Therefore the interval is \(-1 < x \leq 1\).

c) Using the result in a) or by direct computation, find the Maclaurin series for \( \ln(1 - x/2) \).

\[ \text{Ans} \]
\[ \text{Change } x \text{ to } -x/2 \text{ in the Maclaurin series.} \]

Please look at Examples 2 and 6 in section 9.

d) Write down an infinite series that represents the value of \( \ln(2) \). If the first 10 terms of this series are used to approximate the value of \( \ln(2) \), write down an expression that estimates the error committed.
There are two ways to do it. One way is to plug is $x = 1$, you get the alternating harmonic series $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

In this case, the error by the alternating series error estimate is $1/11$.

Another way is to plug in $x = 1/2$ which is inside the interval of convergence. You get

$$\ln(1/2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{1}{2}\right)^n.$$ 

Since $\ln(1/2) = -\ln(2)$, after simplification, you get

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n2^n}.$$ 

Look at the bottom of Example 6 in 11.9.

This series has positive decreasing terms. An error estimate is then given by the expression $\int_{10}^{\infty} \frac{1}{x^n} dx$. There is no need to evaluate the integral.

II. a) Find the third degree Taylor polynomial of $f(x) = \cos(x)$ at $a = \pi/2$.

**Ans.**

This is almost identical to Example 7 in 11.10. $f(\pi/2) = 0$, $f'(\pi/2) = -1$, $f''(\pi/2) = 0$, $f'''(\pi/2) = 1$.

Thus the third degree Taylor polynomial is

$$\cos(x) = 0 - 1 \times (x - \pi/2) + 0 \times (x - \pi/2)^2 + 1/3! \times (x - \pi/2)^3;$$

i.e. $T_3(x) := -(x - \pi/2) + 1/6(x - \pi)^3$.

b) Evaluate the limit

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$ 

This is a simple review on the Maclaurin series of $e^x$.

See Example 9 in 11.10

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*Hints for some of the practice problems in 11.10*

For #2 in 11.10, I hope you have taken a look at the problem before you peek at the solution here.
There are many ways to answer this question. One way is that \( f'(1) = -0.8 \) according to the way the Taylor series is defined. Now the graph is increasing at \( x = 1 \), therefore the series is a wrong one.

Also \( f(1) = 0.4 \) but the graph shows that \( f(1) \) is close to 1.

#10. The Taylor series expansion of \( x^3 \) at \( a = -1 \) can be obtained by the ’cookbook’ Taylor formula. The answer is identical to

\[
x^3 = (1 + (x - 1))^3 = 1 + 3(x - 1) + 3(x - 1)^2 + (x - 1)^3.
\]

Please use the Taylor formula to do a quick check.

#13 can be done in the same way(s) as in #10.

\[
1/x = \frac{1}{1+(x-1)} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots.
\]

Please check that you get the same answer by the Taylor series formula too.

#41 is very similar to the one we did in Maple lab. The expansion of \( \sin(x^2) \) is an alternating series.
III. The polar graph $r = \cos(2\theta)$ is called a rose.

a) Check that the graph has symmetries with respect to both coordinate axes by using the methods discussed in class/text.

b) Plot the graph of the polar equation by setting up a table of appropriate values of $\theta$ and $r$.

c) Find the area of all the regions INSIDE the polar curve $r = \cos(2\theta)$ and OUTSIDE the circle $r = 1/\sqrt{2}$. You may use the identity: $\cos^2(x) \equiv (1 + \cos(2x))/2$. 
IV. Given the infinite series

\[ \sum_{n=1}^{\infty} \frac{n}{e^n}, \]

use the ratio test AND integral test to prove the convergence of this series. (You need to show both methods here)
V. Evaluate the following integrals. Convert the answer in a) in terms of $x$.

\[ a) \int \frac{x^2}{1 + x^2} \, dx, \quad b) \int_0^1 \tan^{-1}(x) \, dx. \]
VI. A particle travels along a parametric curve given by $x = 2 \cos(t)$, $y = 4 \sin(t)$.

Show that this curve is one of the conic sections.

Give a rough sketch of this graph for $0 \leq t \leq 2\pi$ and compute the foci of this curve.
VII. i) If $P = P(t)$ satisfies the equation $\frac{dP}{dt} = 10 - 2P$, with $P(0) = 2$, solve for $P(t)$ explicitly.

ii) Find by any method covered in class,

$$\lim_{t \to \infty} P(t).$$