Solution to HW #2

13.4

9. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $a\cdot(b\times c)$ Yes, It is a scalar.

(b) $a\times(b\cdot c)$ No. $b\cdot c$ is a scalar, so $a\times(b\cdot c)$ is meaningless, as the cross product is defined only for two vectors.

(c) $a\times(b\times c)$ Yes. The result is a vector.

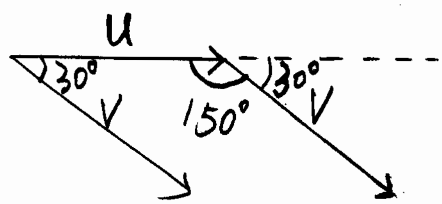
(d) $(a\cdot b)\times c$ No. $a\cdot b$ is a scalar, so the cross product $(a\cdot b)\times c$ is meaningless.

(e) $(a\cdot b)\times(c\cdot d)$ No. Both $a\cdot b$ and $c\cdot d$ are scalars, the cross product $(a\cdot b)\times(c\cdot d)$ is meaningless.

(f) $(a\times b)\cdot(c\times d)$ Yes. The result is a scalar.

11. Find $|u\times v|$ and determine whether $u\times v$ is directed into the page or out of the page.

Solution: If we sketch u and v starting from the same initial point, we see that the angle between them is $30°$. Using Theorem 6, we have



$$|u\times v| = |u||v|\sin 30° = 6\times 8\times \tfrac{1}{2} = 24$$

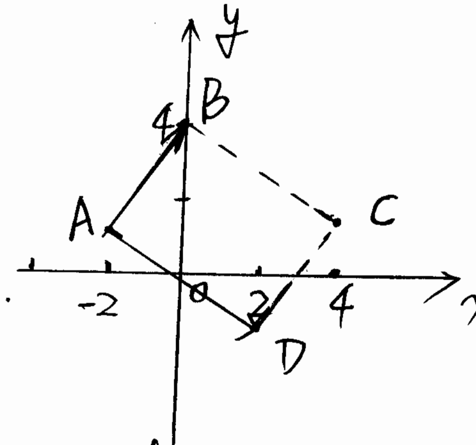
By the right-hand rule, $u\times v$ is directed into the page.

14. If $a=\langle 3,1,2\rangle$, $b=\langle -1,1,0\rangle$, and $c=\langle 0,0,-4\rangle$, show that $a\times(b\times c)\neq(a\times b)\times c$.

proof: $b\times c=\begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 0 & 0 & -4\end{vmatrix}=\begin{vmatrix} 1 & 0 \\ 0 & -4\end{vmatrix}i-\begin{vmatrix} -1 & 0 \\ 0 & -4\end{vmatrix}j+\begin{vmatrix} -1 & 1 \\ 0 & 0\end{vmatrix}k$

$=-4i-4j$

$a\times(b\times c)=\begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ -4 & -4 & 0\end{vmatrix}=8i-8j-8k$



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Similarly, $(a\times b)\times c=8i-8j$.

Thus $a\times(b\times c)\neq(a\times b)\times c$.

23. Find the area of the parallelogram with vertices $A(-2,1)$, $B(0,4)$, $C(4,2)$ and $D(2,-1)$.

Solution: By plotting the vertices, we see that the parallelogram is determined by the vectors $\vec{AB}=\langle 2,3\rangle$, and $\vec{AD}=\langle 2-(-2),-1-1\rangle=\langle 4,-2\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors.

$$\text{Area}=|\vec{AB}\times\vec{AD}|=\left|\begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 4 & -2 & 0\end{vmatrix}\right|=|0i-0j+(-4-12)k|$$

$$=|-16k|=16$$

(In order to compute the cross product, we consider the vector $\vec{AB}$ as the three dimensional vector $\langle 2,3,0\rangle$, similarly for $\vec{AD}$).

28. (a) Find a vector orthogonal to the plane through the points P, Q, and R;

(b) find the area of the triangle PQR.

$P(2,0,-3)$, $Q(3,1,0)$, $R(5,2,2)$

Solution: (a) $\overrightarrow{PQ} = \langle 3-2, 1-0, 0-(-3)\rangle = \langle 1, 1, 3\rangle$

$\overrightarrow{PR} = \langle 5-2, 2-0, 2-(-3)\rangle = \langle 3, 2, 5\rangle$

So a vector orthogonal to the plane through P, Q, R is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = \langle -1, 4, -1\rangle$$

(b) The parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ has area

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -1, 4, -1\rangle| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2},$$

so the area of triangle PQR is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \times 3\sqrt{2} = \frac{3}{2}\sqrt{2}$$

31. Find the volume of the parallelopiped with adjacent edges PQ, PR and PS.

$P(2,0,-1)$, $Q(4,1,0)$, $R(3,-1,1)$, $S(2,-2,2)$

Solution: $a = \overrightarrow{PQ} = \langle 2, 1, 1\rangle$, $b = \overrightarrow{PR} = \langle 1, -1, 2\rangle$, $\overrightarrow{PS} = c = \langle 0, -2, 3\rangle$.

$$a \cdot (b \times c) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}$$

$$= 2 - 3 - 2 = -3$$

So the volume of the parallelopiped is 3 cubic units.

37. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction <0, 3, −4> at the end of the wrench. Find the magnitude of the force needed to supply 100 J of torque to the bolt.

Solution: Using the notation of the text, $r = \langle 0, 0.3, 0\rangle$ and F has the direction $\langle 0, 3, -4\rangle$. The angle θ between them can be determined by

$$\cos\theta = \frac{\langle 0,0.3,0\rangle \cdot \langle 0,3,-4\rangle}{|\langle 0,0.3,0\rangle| \cdot |\langle 0,3,-4\rangle|} = \frac{0.9}{(0.3)5} = 0.6$$

$$\Rightarrow \theta \approx 53.1^\circ$$

Then $|\tau| = |r||F|\sin\theta \Rightarrow 100 = 0.3|F|\sin 53.1^\circ$

$$\Rightarrow |F| \approx 417 N$$

40. Prove part 6 of Theorem 8, that is

$$a \times (b \times c) = (a\cdot c)b - (a \cdot b)c$$

proof: let $a = \langle a_1, a_2, a_3\rangle$, $b = \langle b_1, b_2, b_3\rangle$, $c = \langle c_1, c_2, c_3\rangle$.

Then $b \times c = \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\rangle$, and

$$a \times (b \times c) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$= \langle a_2$