Homework #1 Solutions

Section 13.1

8) $\overrightarrow{AB} = \langle 2,2,1\rangle$, $\overrightarrow{BC} = \langle 0,-6,3\rangle$, $\overrightarrow{CA} = \langle 2,4,4\rangle$
$\implies |AB|^2 = 9,\ |BC|^2 = 45,\ |CA|^2 = 36$
$\implies |AB|^2 + |CA|^2 = 36 + 9 = 45 = |BC|^2$ (Pythagoreans Theorem)
Therefore ABC is a right triangle. ABC is not ososceles, as no two sides have the same length.

18) Completing squares in the equation gives

$$\begin{aligned} 4(x-1)^2 + 4(y+2)^2 + 4z^2 &= 21 \\ (x-1)^2 + (y+2)^2 + z^2 &= \frac{21}{4} \end{aligned}$$

which is an equation of a sphere with center $(1,-2,0)$ and radius $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

19) (a) Proof: If the midpoint of the line segment from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$ is $Q = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$\begin{aligned} |P_1P_2| &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \\ |P_1Q| &= \sqrt{[\frac{1}{2}(x_1+x_2) - x_1]^2 + [\frac{1}{2}(y_1+y_2) - y_1]^2 + [\frac{1}{2}(z_1+z_2) - z_1]^2} \\ &= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \frac{1}{2}|P_1P_2| \\ |QP_2| &= \sqrt{[x_2 - \frac{1}{2}(x_1+x_2)]^2 + [y_2 - \frac{1}{2}(y_1+y_2)]^2 + [z_2 - \frac{1}{2}(z_1+z_2)]^2} \\ &= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \frac{1}{2}|P_1P_2| \end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1(-\frac{1}{2},1,4)$, $P_2(1,\frac{1}{2},5)$ and $P_3(\frac{5}{2},\frac{3}{2},4)$. (Recall that a median of a triangle is a line segment from a vextex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$\begin{aligned} |AP_2| &= \sqrt{0^2 + (\frac{1}{2} - 2)^2 + (5-3)^2} = \frac{5}{2} \\ |BP_3| &= \sqrt{(\frac{5}{2} + 2)^2 + (\frac{3}{2})^2 + (4-5)^2} = \frac{1}{2}\sqrt{94} \\ |CP_1| &= \sqrt{(-\frac{1}{2} - 4)^2 + (1-1)^2 + (4-5)^2} = \frac{1}{2}\sqrt{85} \end{aligned}$$

21) (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(2,-3,6)$, to the xy-plane, namely 6. Therefore $r = 6$ and

an equation of the sphere is

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 36$$

(b) The radius of this sphere is the distance from its center $(2, -3, 6)$ to the yz-plane, which is 2. Therefore, an equation is

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 4$$

(b) The radius of this sphere is the distance from its center $(2, -3, 6)$ to the xz-plane, which is 3. Therefore, an equation is

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 9$$

30) The inequality $1 \leq x^2+y^2+z^2 \leq 25$ is equivalent to $1 \leq \sqrt{x^2+y^2+z^2} \leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5. This is the set of all points on or between the concentric spheres with radii 1 and 5 and center $(0, 0, 0)$.

32) The equation $x^2 + y^2 = 1$ represents the set of all points in R^3 where $x^2 + y^2 = 1$, a surface that intersects the xy-plane in the circle $x^2 + y^2 = 1$, $z = 0$. Since z can vary, the surface is a circular cylinder of radius 1. Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z-axis.

36) Because the box lies in the first quadrant, each point must comprise only nonnegative coordinates. So inequalities describing the region are $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

40) Consider the points P such that the distance from P to $A\,(-1, 5, 3)$ is twice the distance from P to $B\,(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

Solution: Let $P = (x, y, z)$. Then $2|PB| = |PA|$, which implies that $4|PB|^2 = |PA|^2$.

$$\begin{aligned}
4[(x-6)^2 + (y-2)^2 + (z+2)^2] &= (x+1)^2 + (y-5)^2 + (z-3)^2 \\
3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z &= 35 - 144 - 16 - 16 \\
x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z &= -\frac{141}{3}
\end{aligned}$$

By completing the square three times we get

$$(x - \frac{25}{3})^2 + (y-1)^2 + (z + \frac{11}{3})^2 = \frac{332}{9}$$

which is an equation of a sphere with center $(\frac{25}{3}, 1, -\frac{11}{3})$ and radius $\frac{\sqrt{332}}{3}$.

Section 13.2

18) $|\mathbf{a}| = \sqrt{13}$, $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{a} - \mathbf{b} = \mathbf{i} - 8\mathbf{j}$

$2\mathbf{a} = 4\mathbf{i} - 6\mathbf{j}$, $3\mathbf{a} + 4\mathbf{b} = 10\mathbf{i} + 11\mathbf{j}$

26) $|\langle -2, 4, 2\rangle| = \sqrt{24} = 2\sqrt{6}$, so a unit vector in the direction of $\langle -2, 4, 2\rangle$ is

$$\mathbf{u} = \frac{1}{2\sqrt{6}}\langle -2, 4, 2\rangle$$

A vector in the same direction but with length 6 is

$$6\mathbf{u} = 6 \cdot \frac{1}{2\sqrt{6}}\langle -2, 4, 2\rangle = \langle -\frac{6}{\sqrt{6}}, \frac{12}{\sqrt{6}}, \frac{6}{\sqrt{6}}\rangle \quad \text{or} \quad \langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6}\rangle$$

29) $|\mathbf{F}_1 = 10\,lb$ and $|\mathbf{F}_2| = 12\,lb$

$$\begin{aligned}
\mathbf{F}_1 &= -|\mathbf{F}_1|\cos 45^o\mathbf{i} + |\mathbf{F}_1 \sin 45^o\mathbf{j} = -5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j} \\
\mathbf{F}_2 &= 6\sqrt{3}\mathbf{i} + 6\mathbf{j} \\
\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 = (6\sqrt{3} - 5\sqrt{2})\mathbf{i} + (6 + 5\sqrt{2})\mathbf{j} = 3.32\mathbf{i} + 13.07\mathbf{j}
\end{aligned}$$

$|\mathbf{F}| = \sqrt{(3.32)^2 + (13.07)^2} = 13.5\,lb$ $\quad \tan\theta = \frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}}$, so $\theta = \tan^{-1}\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}} = 76^o$.

33) Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write

$$\mathbf{T}_1 = -a\mathbf{i} + b\mathbf{j} \quad \mathbf{T}_2 = a\mathbf{i} + b\mathbf{j}\ (a, b > 0)$$

By similar angles,

$$\frac{b}{a} = \frac{0.08}{4}$$

which implies that $a = 50b$. The force due to gravity is $\mathbf{w} = -0.8 \times 9.8\mathbf{j} = 7.84\mathbf{j}$.

$$\begin{aligned}
& \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \\
\Longrightarrow\ & (-a\mathbf{i} + b\mathbf{j}) + (a\mathbf{i} + b\mathbf{j}) = 7.84\mathbf{j} \\
\Longrightarrow\ & (-50b\mathbf{i} + b\mathbf{j}) + (50\mathbf{i} + b\mathbf{j}) = 2b\mathbf{j} = 7.84\mathbf{j} \\
\Longrightarrow\ & b = \frac{7.84}{2} = 3.92 \quad a = 50b = 196
\end{aligned}$$

This the tensions are

$$\mathbf{T}_1 = -a\mathbf{i} + b\mathbf{j} = -196\,\mathbf{i} + 3.92\,\mathbf{j} \quad \mathbf{T}_2 = a\mathbf{i} + b\mathbf{j} = 196\,\mathbf{i} + 3.92\,\mathbf{j}$$