GUIDELINES FOR PARTIAL FRACTION DECOMPOSITION

Given two polynomials, P and Q, the problem of computing the integral

$$\int \frac{P(x)}{Q(x)}\, dx$$

can be simplified *provided* it is possible to find a factorization of the polynomial Q. While this task may be difficult, it is in theory possible to factor any polynomial with real coefficients into a product of linear factors of the form $(x - \alpha)$ and "irreducible" quadratic factors of the form $(x^2 + \alpha x + \beta)$. For example, we can write:

$$x^3 - 2x^2 + x - 2 = (x - 2)(x^2 + 1),$$

which is the product of the linear factor $(x - 2)$ and the irreducible quadratic factor $(x^2 + 1)$. Notice that it is also true that this polynomial can be factored into the product of linear factors as:

$$(x - 2)(x - i)(x + i) \;\text{ where }\; i = \sqrt{-1}$$

but here we are only interested in factors with *real* coefficients.

The general facts are the following:

Theorem: *Every polynomial with real coefficients may be decomposed into a product of linear and quadratic factors in such a way that each of the factors has real coefficients.*

and

Theorem: *Suppose that two polynomials*

$$S(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \;\text{ and }\; T(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n,$$

are equal for all except possibly a finite number of values of x. Then $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

The method of using the partial fraction decomposition is successful **ONLY** for **proper** rational functions. In what follows, we will assume that we are dealing with such a proper function.

If the degree of P is larger than or equal to the degree of Q, apply long division to reduce the problem to that of integrating the sum of a polynomial and a proper rational function.

The decomposition of a proper rational function into the sum of simpler expressions is known as the

Method of Partial Fraction Expansions

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We divide the method into four cases, depending on the nature of the real factors of Q.

Case 1. The denominator Q can be factored into linear factors *all different*:

$$Q(x) = (x - a_1)(x - a_2)\cdots(x - a_r),$$

with no two of the a_i the same.

In this case we can decompose the ratio P/Q so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \cdots + \frac{A_r}{(x - a_r)},$$

where $A_1, A_2, \ldots, A_r$ are appropriately chosen coefficients.

Example 1: If we wish to find the partial fraction decomposition of

$$\frac{(x^2 + 2x + 3)}{(x^3 - x)}$$

we write

$$\frac{(x^2 + 2x + 3)}{(x^3 - x)} = \frac{A_1}{x} + \frac{A_2}{(x - 1)} + \frac{A_3}{(x + 1)},$$

which is an identity for all $x, x \neq 0, 1, -1$ if and only if

$$x^2 + 2x + 3 = A_1(x - 1)(x + 1) + A_2x(x + 1) + A_3x(x - 1).$$

Collecting like powers of x, we have

$$x^2 + 2x + 3 = (A_1 + A_2 + A_3)x^2 + (A_2 - A_3)x - A_1.$$

Using the second theorem above, for the two polynomials to be equal, the coefficients of like powers of x must match so that we have $-A_1 = 3$ by matching the constant terms. Then

$$A_2 - A_3 = 2 \text{ and } -3 + A_2 + A_3 = 1.$$

It follows that the correct partial fraction decomposition is

$$\frac{(x^2 + 2x + 3)}{(x^3 - x)} = \frac{-3}{x} + \frac{3}{(x - 1)} + \frac{1}{(x + 1)}.$$

Case 2. The polynomial Q can be factored into linear factors, some of which are repeated:

$$Q(x) = (x - a_1)^{s_1}(x - a_2)^{s_2}\cdots(x - a_r)^{s_r}.$$

In this case, a factor of the form $(x - a)^q$ gives rise to the terms

$$\frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \frac{A_3}{(x - a)^3} + \cdots + \frac{A_q}{(x - a)^q}.$$

Example: To find the partial fraction decomposition of

$$\frac{(x + 5)}{(x^3 - 3x + 2)}$$

we note that $x^3 - 3x + 2 = (x-1)^2(x+2)$. This is the Case 2 situation and we look for a decomposition in the form

$$\frac{(x+5)}{(x^3 - 3x + 2)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x+2)}.$$

Multiplying both sides through by $(x-1)^2(x+2)$ we get

$$x + 5 = A_1(x-1)(x+2) + A_2(x+2) + A_3(x-1)^2.$$

Following the same procedure as in the previous example leads to $A_1 = -\frac{1}{3}, A_2 = 2$, and $A_3 = \frac{1}{3}$.

Case 3. The polynomial Q can be factored into linear and quadratic factors, and none of the quadratic factors is repeated:

$$Q(x) = (x-a_1)^{s_1}(x-a_2)^{s_2} \cdots (x-a_r)^{s_r}(x^2 + b_1 x + c_1)(x^2 + b_2 x + c_2) \cdots (x^2 + b_\ell x + c_\ell).$$

In this case, *each* unrepeated quadratic factor gives rise to a term of the form

$$\frac{Ax + B}{x^2 + bx + c}$$

Example:

$$\frac{3x^2 + x - 2}{(x-1)(x^2+1)} = \frac{A_1}{(x-1)} + \frac{A_2 x + A_3}{x^2 + 1}$$

or

$$3x^2 + x - 2 = A_1(x^2+1) + (A_2 x + A_3)(x-1) = (A_1 + A_2)x^2 + (A_3 - A_2)x + (A_1 - A_3).$$

which yields $A_1 = 1, A_2 = 2$, and $A_3 = 3$.

Case 4. The polynomial Q can be factored into linear and quadratic factors, and some of the quadratic factors are repeated.

In this case, factors of the form $(x^2 + bx + c)^q$ give rise to terms

$$\frac{A_1 x + A_2}{x^2 + bx + c} + \frac{A_3 x + A_4}{(x^2 + bx + c)^2} + \frac{A_5 x + A_6}{(x^2 + bx + c)^3} + \cdots + \frac{A_{2q-1} x + A_{2q}}{(x^2 + bx + c)^q}.$$

Example:

$$\frac{2x^3 + 3x^2 + x - 1}{(x+1)(x^2 + 2x + 2)^2} = \frac{A_1}{(x+1)} + \frac{A_2 x + A_3}{(x^2 + 2x + 2)} + \frac{A_4 x + A_5}{(x^2 + 2x + 2)^2}$$

which, after extensive (and tedious) algebra, yields $A_1 = -1, A_2 = 1, A_3 = 3, A_4 = -2$, and $A_5 = -3$.