Eighth Lab For M242, Summer 2000

Due on Thursday, July 6, 2000

Topic : Taylor Approximations

The computation of the Taylor polynomial of a function involves computing the higher order derivatives of the function evaluated at the point $x = a$. This can be automated and has been done in Maple through the command `mtalyor`. We show its use below by applying it to compute the Taylor polynomial of some functions.

```
restart:
readlib(mtaylor):       only once in a session, loads program
mtaylor( exp(x), x=0, 7 );  taylor polynomial of exp(x), at x=0, of order 6
mtaylor( sin(x), x=0, 9 );  taylor polynomial of sin(x), at x=0, of order 8
mtaylor( ln(x), x=1, 8 );  taylor polynomial of ln(x), at x=1, of order 7
```

Note that the last parameter in the `mtaylor` command is one larger than the order of the polynomial desired. I presume Maple does this because the error in the approximation $f(x) - P_n(x) = O(x^{n+1})$ is one order higher than the order of the Taylor polynomial.

The Taylor series for $ln(1 + x)$ and $ln(1 - x)$ at $a = 0$ are

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

and these series are convergent only when $-1 < x < 1$. So these series (only one of them is needed) can be used to compute $\ln t$ only for numbers t in the interval $(0, 2)$ (if x is in $(-1, 1)$ then $1 + x$ is in $(0, 2)$).

To compute the natural logarithm of numbers outside this interval use the following trick. Subtracting the second series from the first we have the series

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots$$

with the series still only convergent when $-1 < x < 1$. However, when x varies over the interval $(-1, 1)$ then $(1 + x)/(1 - x)$ varies over $(0, \infty)$. Hence this series may be used to compute the natural logarithm of any number.

For example to compute $\ln 3$ we find x so that $(1 + x)/(1 - x) = 3$. Solving this equation we get $x = 1/2$ - note this x is in $(-1, 1)$. Hence

$$\ln 3 = \ln\left(\frac{1+x}{1-x}\right)\Bigg|_{x=1/2} = 2\frac{1}{2} + \frac{2}{3}\frac{1}{2^3} + \frac{2}{5}\frac{1}{2^5} + \cdots$$

and we can truncate this series to get an approximate value of $\ln 3$.

Problem 1

1. If we want to compute $\ln t$ for numbers t between 1 and 100 then over what interval will the x range where $t = (1 + x)/(1 - x)$.

2. Use the `mtaylor` command to compute the Taylor polynomials of $\ln((1 + x)/(1 - x))$, $a = 0$, of various orders and plot the polynomial and the original function over the x interval obtained in the previous problem. At what order are the two graphs indistinguishable? You may find it useful to define the procedure generating the Taylor polynomial of order n

```
P := proc(n) mtaylor( ln( (1+x)/(1-x)), x=0, n+1 )  end;
f := ln( (1+x)/(1-x) );
```

   and a name for the original expression. Try `P(4), P(7)`.

3. Now we compute the percentage error between the original and the approximation.

```
E := proc(n) 100*abs( f - P(n) )/abs(f) end;
```

   Plot the percentage error over the interval found in the first part for various values of n. For what value of n will the percentage error over this interval be less than one percent.

4. Use the Taylor polynomial of order n (n from part 3) to compute $\ln 75$ and then check by using Maple that your approximation is accurate to one percent relative error.

Problem 2

The previous problem obtained a polynomial approximation with error less than a desired amount. However, that procedure is not very useful since, it needed the graph of $ln((1+x)/(1+x))$ and hence the value of the logarithm is needed at many points. Below we follow a procedure which does not need values of the function to be approximated at many points.

1. By estimating the remainder, as done in class, what n should be chosen so that $P_n(x)$ approximates $\cos(x)$ with error less than 10^{-8} over the interval $[-\pi, \pi]$. You may assume that $3.1 \leq \pi \leq 3.2$.

2. Using this approximation, compute $\cos(2)$, and compare with the answer provided by Maple. Is the error less than 10^{-8}.

What To Hand In

Submit your handwritten work and the Maple work for Problems 1 and 2. The Maple report must be organized as explained in section 2.1 of the tutorial.

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