Exam #2
Math 243

This is exam #2 of the Spring 2004 semester. You must show all of your work and be sure to clearly indicate your final answer. Good luck!

(1) (20 points) If $f(x, y) = \cos(x^2 + 3xy + 2y^2)$, compute ∇f, f_{xy}, f_{xx}, and f_{yy}.
Easy to use Maple and the diff command. You should find

$$f_{xx} = -\cos(x^2 + 3xy + 2y^2)(2x + 3y)^2 - 2\sin(x^2 + 3xy + 2y^2) \tag{1}$$

$$f_{yy} = -\cos(x^2 + 3xy + 2y^2)(3x + 4y)^2 - 4\sin(x^2 + 3xy + 2y^2) \tag{2}$$

$$f_{xy} = -\cos(x^2 + 3xy + 2y^2)(2x + 3y)(3x + 4y) - 3\sin(x^2 + 3xy + 2y^2) \tag{3}$$

$$\nabla f =< -\sin(x^2 + 3xy + 2y^2)(2x + 3y), -\sin(x^2 + 3xy + 2y^2)(3x + 4y) > \tag{4}$$

(2) (20 points) Consider the curve $x = t$, $y = 4t^{5/2}$, $z = t^2$. Find the curvature at the point $(1, 4, 1)$. Find the arclength of the curve as the parameter t varies from 0 to 1.

We have that

$$\vec{r}(t) = < t, 4t^{5/2}, t^2 > \tag{5}$$

$$\vec{r}'(t) = < 1, 10t^{3/2}, 2t > \tag{6}$$

$$\vec{r}''(t) = < 0, 15t^{1/2}, 2 > \tag{7}$$

hence the curvature can be computed as

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{\sqrt{329}}{(105)^{3/2}} \tag{8}$$

and the arclength as

$$L = \int_0^1 |\vec{r}'(t)| dt \approx 4.38 \tag{9}$$

(3) (20 points) Find the maximum value of the function $f(x, y) = 4xy - 2x^2 - y^4$ on the domain $[-2, 2] \times [-2, 2]$. (Be sure to consider points on the boundary!)

Taking partial derivatives and setting them to zero we find the only critical points are at $(1, 1)$ and $(-1, -1)$. Using the second derivative test these are both local max's. Check the boundary to find that no point on the boundary is higher. For example on the boundary where $x = 2$ we have $f(2, y) = 8y - 8 - y^4$. Use Calc I technique to find max, show no bigger than internal max.

(4) (20 points) Find the linearization of the function $f(x,y) = \sqrt{20 - x^2 - y^2}$ at the point $(1,1)$ and use it to approximate $f(1.1, 0.99)$.

The linearization is

$$f(x,y) \approx \sqrt{18} - \frac{x-1}{\sqrt{18}} - \frac{y-1}{\sqrt{18}} \tag{10}$$

Use to compute $f(1.1, 0.99)$.

(5) (20 points) Find the point on the surface $(x-y)^2 - z^2 = 1$ that is closest to the origin.

The squared distance to the origin is $D = x^2 + y^2 + z^2$. Minimize subject to the constraint that you stay on the surface. Use Lagrange multipliers and consider

$$f = x^2 + y^2 + z^2 + \lambda(x-y)^2 - \lambda z^2 \tag{11}$$

Compute ∇f and equate to zero yields the points $(0, 0, \pm i)$ which we discard, and $(1/2, -1/2, 0)$, $(-1/2, 1/2, 0)$ which we check directly by computing the distance. The distance is the same for each point, hence both points are solutions.