The Elastica

In class we found that the eigenfunctions of the elastica equation:

\[
y'' + \left[ \lambda - \frac{1}{2} \int_0^1 (y')^2 \, dx \right] = 0, \quad y(0) = y(1) = 0, \quad (1)
\]

were given by

\[
y_n = a_n \sin(n\pi x), \quad \frac{a_n^2}{4} = \frac{\lambda}{n^2 \pi^2} - 1, \quad n \geq 1,
\]

\[
y_0 = 0.
\]

A stability diagram is shown below. Even though our stability analysis was NOT correct for the elastica, it did yield the correct classification of the eigenfunctions.

Eigenfunctions of (1). Stable states are thick solid lines.