The Hopf Bifurcation

In class, we discussed the solutions of the following system:

\[
\begin{align*}
\dot{x} &= \alpha x - \omega y - x(x^2 + y^2), \\
\dot{y} &= \omega x + \alpha y - y(x^2 + y^2).
\end{align*}
\]

We noted that the sign of \(\alpha\) determined whether there was a stable limit cycle outside the origin, and the sign of \(\omega\) determined whether the trajectories spun clockwise or counterclockwise.

\[\alpha = -1, \ \omega = 1. \text{ No limit cycle; counterclockwise rotation.}\]

This figure shows the case where \(\alpha\) is negative. In this case, the origin is a stable spiral. \(\omega\) is positive here, so the trajectories spin inward in a counterclockwise direction.
\( \alpha = 1, \omega = -1 \). Limit cycle with clockwise rotation.

This figure shows the case where \( \alpha \) is positive. In this case, the origin is an unstable spiral and there is a stable limit cycle at \( r = 1 \). \( \omega \) is negative here, so the trajectories spin in a clockwise direction.