Hamiltonian Systems

In class, we discussed the solutions of the following system:

\[ \dot{x} = \frac{\partial H}{\partial y}, \]
\[ \dot{y} = -\frac{\partial H}{\partial x}, \]

and noted that \( H \), the Hamiltonian function for the system, was constant along trajectories.

First we consider the case where

\[ \dot{x} = 2y \quad \dot{y} = -8x \quad \implies \quad H = 4x^2 + y^2 + C. \]

The only fixed point is at the origin, where the relevant matrix is

\[ A = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \quad \implies \quad \lambda = \pm 4i. \]

Therefore, the origin is a center.

\[ H = 4x^2 + y^2 + 3 \] with contour lines and level curves.

This figure shows the Hamiltonian function with contours in 3-D as well as the level curves projected onto the \( xy \)-plane to show the trajectories.
Next we consider the case where
\[
\begin{align*}
\dot{x} &= 2x \\
\dot{y} &= -2y
\end{align*}
\implies H = 2xy + C.
\]
The only fixed point is at the origin, where the relevant matrix is
\[
A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \implies \lambda = \pm 2.
\]
Therefore, the origin is a saddle point.

\[H = 2xy + 3\] with contour lines and level curves.

This figure shows the Hamiltonian function with contours in 3-D as well as the level curves projected onto the \(xy\)-plane to show the trajectories. Note the saddle shape of the surface near the origin; this is the motivation for the term \textit{saddle point}. 